Semistationary and stationary reflection

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Abstract

We study the relationship between the semistationary reflection principle and stationary reflection principles. We show that for all regular cardinals $\lambda \geq \omega_2$ the semistationary reflection principle in the space $[\lambda]^{\omega}$ implies that every stationary subset of $E_\lambda := \{ \alpha \in \lambda \mid \text{cf}(\alpha) = \omega \}$ reflects. We also show that for all cardinals $\lambda \geq \omega_3$ the semistationary reflection principle in $[\lambda]^{\omega}$ does not imply the stationary reflection principle in $[\lambda]^{\omega}$.

1 Introduction

In this paper we compare the semistationary reflection principle with stationary reflection principles. The notion of semistationary sets and the semistationary reflection principle were introduced by Shelah [10](Ch.XIII §1). They are closely related to the semiproperness of posets. We review this:

Notation 1.1. For countable sets $x$ and $y$, we write $x \sqsubseteq y$ if $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$.

Definition 1.2 (Shelah [10] Ch.XIII, §1, 1.1.Def.). Let $W$ be a set with $W \supseteq \omega_1$. A subset $S \subseteq [W]^{\omega}$ is called semistationary if the set $\{ y \in [W]^{\omega} \mid (\exists x \in S) x \sqsubseteq y \}$ is stationary in $[W]^{\omega}$.

Definition 1.3 (Shelah [10] Ch.XIII, §1, 1.5.Def.). For a cardinal $\lambda \geq \omega_2$, SSR($[\lambda]^{\omega}$), the semistationary reflection principle in $[\lambda]^{\omega}$, is the following:

SSR($[\lambda]^{\omega}$) $\equiv$ For every semistationary $S \subseteq [\lambda]^{\omega}$, there exists $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^{\omega}$ is semistationary in $[W]^{\omega}$.

In [10](Ch.XIII, §1, 1.4.Claim) Shelah shows that a poset $\mathbb{P}$ is semiproper if and only if $\mathbb{P}$ preserves $\omega_1$ and preserves semistationary subsets of $[\lambda]^{\omega}$ for every $\lambda$. He also shows that (†) holds if and only if SSR($[\lambda]^{\omega}$) holds for every $\lambda \geq \omega_2$. Here (†) is the principle, introduced in Foreman-Magidor-Shelah [3], that every poset preserving stationary subset of $\omega_1$ is semiproper. This is known

MSC (2000) : Primary 03E55, Secondary 03E35
Key words : Semistationary reflection, Stationary reflection.

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to have interesting consequences. Shelah [10](Ch.XII, §2) shows that if Namba forcing is semiproper then (a strong form of) Chang’s conjecture holds. Hence (†) implies Chang’s conjecture. Also, Foreman-Magidor-Shelah [3] shows that (†) implies precipitousness of the nonstationary ideal over $\omega_1$.

In this paper we compare the semistationary reflection principle with the stationary reflection principles defined below. For a regular cardinal $\lambda$, $E^\lambda_\omega$ denotes the set $\{\alpha \in \lambda \mid \text{cf}(\alpha) = \omega\}$.

**Definition 1.4.** For a regular cardinal $\lambda \geq \omega_2$, let $\text{SR}(\lambda)$ denote the following stationary reflection principle:

$$\text{SR}(\lambda) \equiv \text{For every stationary } B \subseteq E^\lambda_\omega, \text{ there exists } \gamma < \lambda \text{ such that } B \cap \gamma \text{ is stationary.}$$

For a cardinal $\lambda \geq \omega_2$, let $\text{SR}([\lambda]^\omega)$ denote the stationary reflection principle in $[\lambda]^\omega$:

$$\text{SR}([\lambda]^\omega) \equiv \text{For every stationary } S \subseteq [\lambda]^\omega, \text{ there exists } W \subseteq \lambda \text{ such that } |W| = \omega_1 \subseteq W \text{ and } S \cap [W]^\omega \text{ is stationary in } [W]^\omega.$$  

It is easy to see that $\text{SR}([\lambda]^\omega)$ implies $\text{SSR}([\lambda]^\omega)$. (See Section 2.2.)

Our main results are as follows:

**Theorem 1.5.** Let $\lambda$ be a regular cardinal $\geq \omega_2$. Then $\text{SSR}([\lambda]^\omega)$ implies $\text{SR}(\lambda)$.

**Theorem 1.6.** If $\kappa$ is a supercompact cardinal, then there exists a generic extension in which $\text{SSR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ but $\text{SR}([\lambda]^\omega)$ does not hold for any $\lambda \geq \omega_3$.

Foreman-Magidor-Shelah [3] shows that if $\text{SR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ then (†) holds. Theorem 1.6 claims that the converse is not true. Also, as we prove in Section 5, $\text{SSR}(\omega_2^\omega)$ implies $\text{SR}(\omega_2^\omega)$. Theorem 1.6 is optimal in this sense.

This paper is organized as follows: In Section 2 we discuss some preliminaries for this paper. In Section 3 we present a certain type of stationary subset of $[\lambda]^\omega$ which was first introduced by Shelah. This type of stationary set plays a central role in the proofs of both Theorems 1.5 and 1.6. In Section 4 we prove Theorem 1.5. In Section 5 we compare $\text{SSR}([\lambda]^\omega)$ and $\text{SR}([\lambda]^\omega)$. Among other things, we prove Theorem 1.6.

# 2 Preliminaries

## 2.1 Notations

We follow the notations of Jech [4]. Here we present those which may not be general.

For a regular cardinal $\gamma$ and an inaccessible cardinal $\kappa$, let $\text{Col}(\gamma, < \kappa)$ denote the Lévy collapse which forces $\kappa$ to be $\gamma^+$. 

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For a regular cardinal $\gamma$ and a limit ordinal $\delta > \gamma$, let $E_\gamma^\delta$ denote the set $\{ \alpha \in \delta \mid \text{cf}(\alpha) = \gamma \}$. Note that if $\text{cf}(\delta) > \gamma$ then $E_\gamma^\delta$ is stationary in $\delta$.

Fact 2.2 be sets with a greatest element. This makes our definitions and arguments slightly simpler.

For a set $x$ of ordinals let

$$\sup x := \sup\{\alpha + 1 \mid \alpha \in x\}.$$ 

In this paper we use $\sup$ rather than $\sup$. We are mainly interested in sets of ordinals which do not have a greatest element. For such $x$, $\sup x = \sup x$. The merit of using $\sup$ is that $\sup x$ is a limit ordinal if and only if $x$ does not have a greatest element. This makes our definitions and arguments slightly simpler.

2.2 Basics on stationary sets and semistationary sets

For basics on the notion of club or stationary subsets of $\mathcal{P}_\kappa W$ consult Jech [4]. When $\kappa = \omega_1$, we prefer to use $[W]^\omega$ rather than $\mathcal{P}_\kappa W$. A subset of $[W]^\omega$ is said to be club (stationary) if it is club (stationary) in $\mathcal{P}_\kappa W$. This paper uses the following two facts without any reference:

Fact 2.1 (Kueker [6]). Let $\kappa$ be a regular uncountable cardinal, $W$ be a set with $\kappa \subseteq W$, and let $C \subseteq \mathcal{P}_\kappa W$ be a club. Then there exists a function $f : [W]^\omega \rightarrow W$ such that $\{ x \in \mathcal{P}_\kappa W \mid f^n[x]^\omega \subseteq x \land x \cap \kappa \subseteq C \} \subseteq C$. If $\kappa = \omega_1$ then there exists a function $f : [W]^\omega \rightarrow W$ such that $\{ x \in \mathcal{P}_\kappa W \mid f^n[x]^\omega \subseteq x \} \subseteq C$.

Fact 2.2 (Menas [9]). Let $\kappa$ be a regular uncountable cardinal, and let $W$ and $\bar{W}$ be sets with $\kappa \subseteq W \subseteq \bar{W}$.

1. If $C \subseteq \mathcal{P}_\kappa W$ is a club then the set $\{ \bar{x} \in \mathcal{P}_\kappa W \mid \bar{x} \cap W \subseteq C \}$ is a club in $\mathcal{P}_\kappa W$. Hence if $\bar{S} \subseteq \mathcal{P}_\kappa W$ is stationary then the set $\{ \bar{x} \cap W \mid \bar{x} \in \bar{S} \}$ is stationary in $\mathcal{P}_\kappa W$.

2. If $\bar{C} \subseteq \mathcal{P}_\kappa W$ is a club then the set $\{ \bar{x} \cap W \mid \bar{x} \in \bar{C} \}$ contains a club in $\mathcal{P}_\kappa W$. Hence if $S \subseteq \mathcal{P}_\kappa W$ is stationary then the set $\{ \bar{x} \in \mathcal{P}_\kappa W \mid \bar{x} \cap W \in S \}$ is stationary in $\mathcal{P}_\kappa W$.

Basics on semistationary subsets of $[W]^\omega$ was studied in Shelah [10] (Ch.XIII, §1). The following lemma is an analogy of Fact 2.2 for semistationary sets. In the case of (2), a stronger result holds. Part (2) of the following lemma illustrates a unique property of semistationary sets:

Lemma 2.3. Let $W$ and $\bar{W}$ be sets with $\omega_1 \subseteq W \subseteq \bar{W}$.

1. If $\bar{S} \subseteq [\bar{W}]^\omega$ is semistationary then the set $\{ \bar{x} \cap W \mid \bar{x} \in \bar{S} \}$ is semistationary.

2. If $S \subseteq [W]^\omega$ is semistationary then $S$ is also semistationary in $[\bar{W}]^\omega$.

Proof. (1) is clear from Fact 2.2 (1). We prove (2).

Suppose that $S \subseteq [W]^\omega$ is semistationary. Let $T := \{ y \in [W]^\omega \mid (\exists x \in S) \ x \subseteq y \}$ and $\bar{T} := \{ \bar{y} \in [\bar{W}]^\omega \mid (\exists \bar{x} \in \bar{S}) \ \bar{x} \subseteq \bar{y} \}$. Then $T$ is stationary in $[W]^\omega$, and $\bar{T} = \{ \bar{y} \in [\bar{W}]^\omega \mid \bar{y} \cap W \in T \}$. Hence $\bar{T}$ is stationary in $[\bar{W}]^\omega$. Therefore $S$ is semistationary in $[W]^\omega$.  \[\square\]
2.3 Basics on reflection principles

In this paper we use the following reflection principles which are generalizations of SR(\([\lambda]^\omega\)) and SSR(\([\lambda]^\omega\)):

**Definition 2.4.** For a cardinal \(\lambda \geq \omega_2\) and a regular cardinal \(\kappa\) with \(\omega_2 \leq \kappa \leq \lambda\), let SSR(\([\lambda]^\omega\), \(<\kappa\)) and SR(\([\lambda]^\omega\), \(<\kappa\)) be the following reflection principles:

\[
\text{SR}(\([\lambda]^\omega\), \(<\kappa\)) \equiv \text{For every stationary } S \subseteq [\lambda]^\omega, \text{ there exists } W \in \mathcal{P}_\kappa \lambda \text{ such that } \omega_1 \subseteq W \cap \kappa \in \kappa \text{ and } S \cap [W]^\omega \text{ is stationary.}
\]

\[
\text{SSR}(\([\lambda]^\omega\), \(<\kappa\)) \equiv \text{For every semistationary } S \subseteq [\lambda]^\omega, \text{ there exists } W \in \mathcal{P}_\kappa \lambda \text{ such that } \omega_1 \subseteq W \cap \kappa \in \kappa \text{ and } S \cap [W]^\omega \text{ is semistationary.}
\]

Here we review basics about the above reflection principles. First we observe that these are generalizations of SR(\([\lambda]^\omega\)) and SSR(\([\lambda]^\omega\)):

**Lemma 2.5.** Let \(\lambda\) be a cardinal \(\geq \omega_2\).

1. SR(\([\lambda]^\omega\)) is equivalent to SR(\([\lambda]^\omega\), \(<\omega_2\)).
2. SSR(\([\lambda]^\omega\)) is equivalent to SSR(\([\lambda]^\omega\), \(<\omega_2\)).

**Proof.** (1). It suffices to show that SR(\([\lambda]^\omega\)) implies SR(\([\lambda]^\omega\), \(<\omega_2\)). Before starting, take a surjection \(\pi_\alpha : \omega_1 \to \alpha\) for each \(\alpha < \omega_2\) and let \(f : \omega_2 \times \omega_1 \to \omega_2\) be the function defined by \(f(\alpha, \xi) = \pi_\alpha(\xi)\) for each \(\langle \alpha, \xi\rangle \in \omega_2 \times \omega_1\).

Assume that SR(\([\lambda]^\omega\)) holds. To show that SR(\([\lambda]^\omega\), \(<\omega_2\)) holds, take an arbitrary stationary \(S \subseteq [\lambda]^\omega\). We may assume that every element of \(S\) is closed under \(f\). By SR(\([\lambda]^\omega\)), we may choose a \(W \subseteq \lambda\) such that \([W] = \omega_1 \subseteq W\) and \(S \cap [W]^\omega\) is stationary. Note that \(W\) is closed under \(f\) because stationary many elements of \([W]^\omega\) are closed under \(f\). Because \(\omega_1 \subseteq W\), if \(\alpha \in W \cap \omega_2\) then \(\alpha \subseteq W\). Hence \(W \cap \omega_2 \subseteq \omega_2\). Therefore \(W\) witnesses that SR(\([\lambda]^\omega\), \(<\omega_2\)) holds for \(S\).

(2). It suffices to show that SSR(\([\lambda]^\omega\)) implies SSR(\([\lambda]^\omega\), \(<\omega_2\)). Assume that SSR(\([\lambda]^\omega\)) holds. Take an arbitrary semistationary \(S \subseteq [\lambda]^\omega\). Let \(W \subseteq \lambda\) be a witness of SSR(\([\lambda]^\omega\)) for \(S\) and let \(W' := W \cup \text{sup}(W \cap \omega_2)\). Then \(\omega_1 \subseteq W' \cap \omega_2 \subseteq \omega_2\). Moreover \(S \cap [W']^\omega\) is semistationary in \([W']^\omega\) by Lemma 2.3. Hence \(S \cap [W']^\omega\) is semistationary. Therefore \(W'\) witnesses SSR(\([\lambda]^\omega\), \(<\omega_2\)) for \(S\). \(\square\)

Next we observe that SR(\([\lambda]^\omega\), \(<\kappa\)) implies SSR(\([\lambda]^\omega\), \(<\kappa\))

**Lemma 2.6.** Let \(\lambda\) be a cardinal \(\geq \omega_2\) and \(\kappa\) be a regular cardinal with \(\omega_2 \leq \kappa \leq \lambda\). Then SSR(\([\lambda]^\omega\), \(<\kappa\)) is equivalent to the following principle:

\[
\text{SSR'}(\([\lambda]^\omega\), \(<\kappa\)) \equiv \text{For every stationary } S \subseteq [\lambda]^\omega, \text{ there exists } W \in \mathcal{P}_\kappa \lambda \text{ such that } \omega_1 \subseteq W \cap \kappa \in \kappa \text{ and } S \cap [W]^\omega \text{ is semistationary.}
\]

Therefore SR(\([\lambda]^\omega\), \(<\kappa\)) implies SSR(\([\lambda]^\omega\), \(<\kappa\)).
Proof. It suffices to show that SSR' implies SSR. Assume that SSR'([\lambda]^\omega, < \kappa) holds. Take an arbitrary semistationary \( S \subseteq [\lambda]^\omega \). Then \( T := \{ y \in [\lambda]^\omega \mid (\exists x \in S) x \subseteq y \} \) is stationary. Let \( W \subseteq \lambda \) be a witness of SSR'([\lambda]^\omega, < \kappa) for \( T \). Here note that
\[
\{ y \in [W]^\omega \mid (\exists x \in T \cap [W]^\omega) x \subseteq y \} = \{ y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \subseteq y \}.
\]
Hence \( \{ y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \subseteq y \} \) is stationary and thus \( S \cap [W]^\omega \) is semistationary. Therefore \( W \) witnesses SSR([\lambda]^\omega, < \kappa) for \( S \).

The following is very easy:

Lemma 2.7. Let \( \lambda \) and \( \lambda' \) be cardinals and let \( \kappa \) and \( \kappa' \) be regular cardinals such that \( \omega_2 \leq \kappa \leq \kappa' \leq \lambda \).

1. SR([\lambda]^\omega, < \kappa) implies SR([\lambda']^\omega, < \kappa).
2. SSR([\lambda]^\omega, < \kappa) implies SSR([\lambda']^\omega, < \kappa').

Proof. (1) Assume that SR([\lambda]^\omega, < \kappa) holds. Take an arbitrary stationary \( S' \subseteq [\lambda']^\omega \). Then \( S := \{ x \in [\lambda]^\omega \mid x \cap \lambda' \subseteq S' \} \) is stationary. Hence there exists \( W \in P_\kappa \lambda \) such that \( \omega_1 \subseteq W \cap \kappa \subseteq \kappa \) and \( S \cap [W]^\omega \) is stationary. Let \( W' := W \cap \lambda' \). Then \( W' \in P_\kappa \lambda' \) and \( \omega_1 \subseteq W' \cap \kappa \subseteq \kappa \). Moreover \( S' \cap [W']^\omega = \{ x \cap W' \mid x \in S \cap [W]^\omega \} \). Thus \( S' \cap [W']^\omega \) is stationary. Therefore \( W' \) witnesses SSR([\lambda']^\omega, < \kappa) for \( S' \).

(2) Assume that SSR([\lambda']^\omega, < \kappa) holds. Take an arbitrary semistationary \( S' \subseteq [\lambda']^\omega \). Let \( S, W \) and \( W' \) be as in (1). Then, using Lemma 2.3, the same argument as (1) shows that \( S' \cap [W']^\omega \) is semistationary. Let \( W'' := W' \cup \sup(W' \cap \kappa') \). Then \( W'' \in P_\kappa \lambda' \) and \( \omega_1 \subseteq W'' \cap \kappa' \subseteq \kappa' \). Moreover \( S' \cap [W'']^\omega \) is semistationary in \( [W'']^\omega \) by Lemma 2.3. Hence \( S' \cap [W'']^\omega \) is semistationary. Therefore \( W'' \) witnesses SSR([\lambda']^\omega, < \kappa') for \( S' \).

We end this section with the following:

Lemma 2.8. Let \( \lambda \) be a cardinal \( \geq \omega_2 \) and \( \kappa \) be a regular cardinal with \( \omega_2 \leq \kappa \leq \lambda \).

1. ([Feng-Jech 2]) Assume that SR([\lambda]^\omega, < \kappa) holds. If \( S \subseteq [\lambda]^\omega \) is stationary then the set \( \{ W \in P_\kappa \lambda \mid S \cap [W]^\omega \text{ is stationary} \} \) is stationary in \( P_\kappa \lambda \).

2. Assume that SSR([\lambda]^\omega, < \kappa) holds. If \( S \subseteq [\lambda]^\omega \) is semistationary then the set \( \{ W \in P_\kappa \lambda \mid S \cap [W]^\omega \text{ is semistationary} \} \) is co-bounded, that is, there exists \( W^* \in P_\kappa \lambda \) such that \( S \cap [W]^\omega \text{ is semistationary for every } W \in P_\kappa \lambda \text{ with } W^* \supseteq W \).

Proof. (2) is clear from Lemma 2.3 (2). We prove (1).

Take an arbitrary stationary \( S \subseteq [\lambda]^\omega \) and an arbitrary function \( f : [\lambda]^\omega \to \lambda \). It suffices to find a \( W \in P_\kappa \lambda \) such that \( W \cap \kappa \subseteq \kappa \) and \( W \) is closed under \( f \). Let \( S' \) be the set of all \( x \in S \) closed under \( f \). Then \( S' \) is stationary. Hence there exists \( W \in P_\kappa \lambda \) such that \( \omega_1 \subseteq W \cap \kappa \subseteq \kappa \) and \( S' \cap [W]^\omega \) is stationary. Note that \( W \) is closed under \( f \) because stationary many elements of \( [W]^\omega \) are closed under \( f \). Moreover \( S \cap [W]^\omega \) is stationary because \( S' \subseteq S \). Therefore \( W \) is a desired one.
3 Sup depending stationary sets

Here we present a type of stationary set which plays a central role in the proofs of both Theorems 1.5 and 1.6:

**Lemma 3.1** (The case when \( n = 1 \) is due to Shelah). Suppose that \( 0 < n < \omega \) and that \( \mu_0 < \mu_1 < \cdots < \mu_n \) are regular uncountable cardinals. Moreover, suppose that \( A \subseteq E^n_\omega \) is stationary and that, for each \( m \) with \( 1 \leq m \leq n \), \( \langle A^m_\alpha \mid \alpha \in \mu_{m-1} \rangle \) is a sequence of stationary subsets of \( E^n_\omega \). Let \( S \) be the set of all \( x \in \mathcal{P}_{\mu_0} \mu_n \) such that

1. \( x \cap \mu_0 \in A \),
2. \( \text{sup}(x \cap \mu_m) \in A^m_{\text{sup}(x \cap \mu_{m-1})} \) for each \( m \) with \( 1 \leq m \leq n \).

Then \( S \) is stationary in \( \mathcal{P}_{\mu_0} \mu_n \).

This type of stationary set was considered by Shelah, and in Shelah-Shioya [12] and Shelah [11], such sets are used to obtain consequences of the stationary reflection principle. The proof of the above lemma for the case when \( n = 1 \) can be found in Shelah-Shioya [12]. Although there are no difficulties in generalization, we give the complete proof of Lemma 3.1.

We use the following game \( \mathcal{G} \):

**Definition 3.2.** Suppose that \( 0 < n < \omega \) and that \( \mu_0 < \mu_1 < \cdots < \mu_n \) are regular uncountable cardinals. For an \( \alpha \in \mu_0 \) and a function \( f : [\mu_n]^\omega \rightarrow \mu_n \) let \( \mathcal{G}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f) \) be the following two players game of length \( \omega \):

In the \( k \)-th stage, first Player I plays a \( \langle \beta_m^k \mid 1 \leq m \leq n \rangle \) and then Player II plays a \( \langle \gamma_m^k \mid 1 \leq m \leq n \rangle \) so that \( \beta_k^m \leq \gamma_k^m < \mu_m \) for each \( m \).

<table>
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<th>( \beta_0^1, \ldots, \beta_0^n )</th>
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Player II wins if \( \text{cl}_f(\alpha \cup \{ \gamma_k^m \mid 1 \leq m \leq n \land k \in \omega \}) \cap \mu_0 = \alpha \), where \( \text{cl}_f(x) \) denotes the closure of \( x \) under \( f \). Otherwise Player I wins.

Note that \( \mathcal{G}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f) \) is an open game for Player I. Hence it is determined. The following is a key lemma:

**Lemma 3.3.** Suppose that \( 0 < n < \omega \) and that \( \mu_0 < \mu_1 < \cdots < \mu_n \) are regular uncountable cardinals. Then for every function \( f : [\mu_n]^\omega \rightarrow \mu_n \), there are club many \( \alpha \in \mu_0 \) such that Player II has a winning strategy in \( \mathcal{G}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f) \).

Proof. Take an arbitrary function \( f : [\mu_n]^\omega \rightarrow \mu_n \) and let \( A \) be the set of all \( \alpha \in \mu_0 \) such that Player I has a winning strategy in \( \mathcal{G}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f) \). It suffices to show that \( A \) is nonstationary.

Assume that \( A \) is stationary. For each \( \alpha \in A \), take a winning strategy \( \sigma_\alpha \) for Player I in \( \mathcal{G}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f) \). Let \( \theta \) be a sufficiently large regular cardinal.
Then we can take an elementary submodel $M$ of $(\mathcal{H}, \in, \langle \sigma_\alpha \mid \alpha \in A \rangle)$ such that $\alpha^* := M \cap \mu_0 \in A$.

By induction on $k$, we construct a sequence of moves $(\beta_k^1, \ldots, \beta_k^n)$ for each $k \in \omega$ in $\mathcal{D}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$ so that $\gamma_k^1, \ldots, \gamma_k^n \in M$ for each $k \in \omega$. Suppose that $k \in \omega$ and that $(\beta_k^1, \ldots, \beta_k^n)$ and $(\gamma_k^1, \ldots, \gamma_k^n)$ have been chosen for each $l < k$. Let

$$\langle \beta_k^1, \ldots, \beta_k^n \rangle := \sigma_{\alpha^*} \langle \gamma_1^1, \ldots, \gamma_1^n \mid l < k \rangle,$$

and for each $m$ with $1 \leq m \leq n$, let

$$\gamma_k^m := \sup \{ \pi_m \circ \sigma_\alpha \langle \gamma_1^1, \ldots, \gamma_1^n \mid l < k \rangle \mid \alpha \in A \},$$

where $\pi_m : \mu_1 \times \cdots \times \mu_n \to \mu_m$ is the $m$-th projection. Clearly $\beta_k^m \leq \gamma_k^m$ for each $m$. Note that $\gamma_k^m < \mu_m$ since $\mu_m$ is regular and $A \subseteq \mu_0 < \mu_m$. Note also that $\gamma_k^m \in M$ because $\langle \gamma_1^1, \ldots, \gamma_1^n \mid l < k \rangle \in M$. This completes the induction.

First note that $\langle \beta_k^1, \ldots, \beta_k^n, \gamma_k^1, \ldots, \gamma_k^n \mid k \in \omega \rangle$ is a sequence of moves in $\mathcal{D}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$ in which Player I has played according to winning strategy $\sigma_\alpha$. Hence Player I wins. On the other hand, $\alpha^* \cup \{ \gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega \} \subseteq M$, and $M$ is closed under $f$. Thus $\sup_f (\alpha^* \cup \{ \gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega \}) \subseteq M \cap \mu_0 = \alpha^*$. Therefore $\sup_f (\alpha^* \cup \{ \gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega \}) = \alpha^*$, so that Player II wins with this sequence of moves. This is a contradiction. \hfill \Box

Now we prove Lemma 3.1.

Proof of Lemma 3.1. We prove Lemma 3.1 by induction on $n$. Suppose that $n = 1$ or that $n > 1$ and that the lemma holds for $n - 1$. We prove the lemma for $n$. Take an arbitrary function $f : [\mu_n]^{<\omega} \to \mu_n$. It suffices to find $x^* \in S$ such that $x^*$ is closed under $f$ and $x^* \cap \mu_0 \in \mu_0$.

By Lemma 3.3, there exists $\alpha^* \in A$ such that Player II has a winning strategy $\sigma^*$ in $\mathcal{D}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$.

Let $S'$ be the set of all $y \in P_{\mu_0, \mu_n}$ such that

$(1)$ $y \cap \mu_1 \in A_{\beta_1}^1$,

$(2)$ $\sup(y \cap \mu_n) \in A_{\sup(y \cap \mu_{n-1})}^m$ for each $m$ with $2 \leq m \leq n$.

Then $S'$ is stationary in $P_{\mu_0, \mu_n}$. If $n = 1$ then this is clear. $(2)$ claims nothing. If $n > 1$ then this follows from the lemma for $n - 1$.

Choose $y^* \in S'$ such that $\alpha^* \subseteq y^*$ and such that $y^*$ is closed under $\sigma^*$ and $f$. For each $m$ with $1 \leq m \leq n$, take a cofinal sequence $\langle \beta_k^m \mid k \in \omega \rangle$ in $y^* \cap \mu_m$. Moreover let

$$\langle \gamma_1^1, \ldots, \gamma_1^n \rangle := \sigma^* \langle \beta_1^1, \ldots, \beta_1^n \mid l < k \rangle$$

for each $k \in \omega$. Note that $\langle \gamma_k^m \mid k \in \omega \rangle$ is a cofinal sequence in $y^* \cap \mu_m$ for each $m$ with $1 \leq m \leq n$. This is because $y^*$ is closed under $\sigma^*$. Finally let $x^* := \sup_f (\alpha^* \cup \{ \gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega \})$. We show that $x^* \in S$ and $x^* \cap \mu_0 \in \mu_0$.

First note that $x^* \subseteq y^*$ because $\alpha^* \cup \{ \gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega \} \subseteq y^*$ and $y^*$ is closed under $f$. Then, because $\langle \gamma_k^m \mid k \in \omega \rangle$ is cofinal in $y^* \cap \mu_m$, $\sup(x^* \cap \mu_m) = \sup(y^* \cap \mu_m)$ for each $m$ with $1 \leq m \leq n$. Hence, by (2) above,
Now it follows from (i), (ii) and (iii) that

Thus

Moreover

\[ x^* \in \mathcal{A}'_{\text{sup}(x^* \cap \mu_m)} \] for each \( m \) with \( 2 < m < n \).

Moreover \( x^* \cap \mu_0 = \alpha^* \). This is because \( \sigma^* \) is a winning strategy for Player II in \( \mathcal{D}(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f) \). Also recall that \( \text{sup}(x^* \cap \mu_1) = \text{sup}(y^* \cap \mu_1) \in A^\lambda_1 \).

Thus

(i) \( \text{sup}(x^* \cap \mu_0) \in A \),

(ii) \( \text{sup}(x^* \cap \mu_0) \in A \),

(iii) \( \text{sup}(x^* \cap \mu_1) \in A^\lambda_1 \).

Now it follows from (i), (ii) and (iii) that \( x^* \in S \) and \( x^* \cap \mu_0 \notin \mu_0 \).

This completes the proof. \( \square \)

4 \ SSR(\[ \lambda \]) and SR(\[ \lambda \])

In this section, we prove Theorem 1.5. In fact, we prove the following more general theorem:

**Theorem 4.1.** Let \( \lambda \) and \( \kappa \) be regular cardinals such that \( \omega_2 \leq \kappa \leq \lambda \). Then \( \text{SSR}(\lceil \lambda \rceil^\omega, < \kappa) \) implies \( \text{SR}(\lambda) \).

By Lemma 2.5 and 2.7, Theorem 1.5 follows from Theorem 4.1. Theorem 4.1 can be easily obtained from Lemma 3.1 and the following lemma:

**Lemma 4.2.** Let \( \lambda \) be a cardinal and \( \kappa \) be a regular cardinal such that \( \omega_2 \leq \kappa \leq \lambda \). Assume that \( S \subseteq [\lambda]^\omega \) and that there exists \( W \in \mathcal{P}_\lambda \) such that \( \omega_1 \subseteq W \cap \kappa \subseteq \kappa \) and \( S \cap [W]^{\omega} \) is semistationary. Let \( W^* \in \mathcal{P}_\lambda \) be such that

1. \( \omega_1 \subseteq W^* \cap \kappa \subseteq \kappa \) and \( S \cap [W^*]^{\omega} \) is semistationary,
2. for every \( W \in \mathcal{P}_\lambda \), if \( \omega_1 \subseteq W \cap \kappa \subseteq \kappa \) and \( S \cap [W]^{\omega} \) is semistationary

then \( \text{sup} W^* \leq \text{sup} W \).

Then

\[ S_0 := \{ y \in [W^*]^{\omega} \mid (\exists x \in S \cap [W^*]^{\omega}) \ x \subseteq y \land \text{sup} x = \text{sup} y \} \]

is stationary in \([W^*]^{\omega}\).

**Proof.** Assume that \( S_0 \) is not stationary. Then \( S_1 := \{ y \in [W^*]^{\omega} \mid (\exists x \in S \cap [W^*]^{\omega}) \ x \subseteq y \land \text{sup} x < \text{sup} y \} \) is stationary. For each \( y \in S_1 \), choose \( x_y \in S \) with \( x_y \subseteq y \) and \( \text{sup} x_y < \text{sup} y \) and choose \( \alpha_y \in y \) with \( \text{sup} x_y \leq \alpha_y \).

Then there exists \( \alpha' \in W^* \) such that \( S' := \{ y \in S_1 \mid \alpha_y = \alpha' \} \) is stationary.

Let \( W' := W^* \cap \alpha' \). Clearly \( W' \in \mathcal{P}_\lambda \) and \( \omega_1 \subseteq W' \cap \kappa \subseteq \kappa \). Moreover \( \text{sup} W' < \text{sup} W^* \). Hence if we show that \( S \cap [W']^{\omega} \) is semistationary then this contradicts property (2) of \( W^* \).

First note that if \( y \in S' \) then \( x_y \in [W']^{\omega} \) and \( x_y \subseteq y \cap W' \). Thus

\[ \{ y \in W' \mid y \in S' \} \subseteq \{ z \in [W']^{\omega} \mid (\exists x \in S \cap [W']^{\omega}) \ x \subseteq z \} \]

Now the left side is stationary in \([W']^{\omega}\) because \( S' \) is stationary in \([W^*]^{\omega}\).

Therefore the right side is stationary, that is, \( S \cap [W']^{\omega} \) is semistationary.

This completes the proof. \( \square \)
Proof of Theorem 4.1. Assume that SSR([\lambda]^\omega, < \kappa) holds. Take an arbitrary stationary B \subseteq E_\lambda^\kappa. We show that B reflects.

Take a pairwise disjoint sequence \langle B_\alpha | \alpha \in \omega_1 \rangle of stationary subsets of B. Let S be the set of all x \in [\lambda]^\omega such that x \cap \omega_1 \in \omega_1 and \text{s}up x \in B_{x \cap \omega_1}. Then S is stationary by Lemma 3.1. By Lemma 4.2, there exists W \in \mathcal{P}_\kappa \lambda such that \omega_1 \subseteq W \cap \kappa \in \kappa and S_0 := \{ y \in [W]^\omega | (\exists x \in S \cap [W]^\omega) x \subseteq y \land \text{s}up x = \text{s}up y \} is stationary in [W]^\omega. Here note that S_0 \subseteq S. Hence S \cap [W]^\omega is stationary. We claim that cf(\text{s}up W) = \omega.

Clearly \text{s}up W is a limit ordinal. Assume that cf(\text{s}up W) = \omega. Then C := \{ y \in [W]^\omega | \text{s}up y = \text{s}up W \} is club in [W]^\omega. But if y_1, y_2 \in S \cap C then \text{s}up(y_1 \cap \omega_1) = \text{s}up(y_2 \cap \omega_1) by the construction of S. Hence |\{ y \cap \omega_1 | y \in S_0 \cap C \}| \leq 1. This contradicts \omega_1 \subseteq W and S \cap [W]^\omega is stationary.

Now cf(\text{s}up W) > \omega and S \cap [W]^\omega is stationary. Hence \{ \text{s}up y \ | \ y \in S \cap [W]^\omega \} is stationary in \text{s}up W. Recall that \text{s}up y \in B for every y \in S. Therefore B \cap \text{s}up W is stationary. This completes the proof.

5 SSR([\lambda]^\omega) and SR([\lambda]^\omega)

As we mentioned in Section 1, first we prove that SSR([\omega_2]^\omega) implies SR([\omega_2]^\omega) and thus that they are equivalent. This is essentially proved in Todorcević [13]. After that, we prove Theorem 1.6.

Theorem 5.1. SSR([\omega_2]^\omega) implies SR([\omega_2]^\omega).

Proof. Assume that SSR([\omega_2]^\omega) holds. To show that SR([\omega_2]^\omega) holds, take an arbitrary stationary S \subseteq [\omega_2]^\omega. Fix a bijection \pi_\alpha : \omega_1 \rightarrow \alpha for each \alpha \in [\omega_1, \omega_2).

We may assume that if x \in S then \omega_1 < \text{s}up x and x is closed under \pi_\alpha, \pi^\alpha for each \alpha \in x \setminus \omega_1.

By Lemma 2.5 let \alpha^* be the least \alpha \in \omega_2 such that S \cap [\alpha]^\omega is semistationary. Then let S_0 be the set of all y \in [\alpha^*]^\omega such that

(1) for some x \in S \cap [\alpha^*]^\omega, x \subseteq y and \text{s}up x = \text{s}up y,

(2) y is closed under \pi_\alpha, \pi^\alpha for each \alpha \in y \setminus \omega_1.

By Lemma 4.2, S_0 is stationary in [\alpha^*]^\omega. For each y \in S_0, choose an x_y \in S \cap [\alpha^*]^\omega witnessing (1).

Note that if y \in S_0 then

y \cap \alpha = \pi_\alpha^\omega(y \cap \omega_1) = \pi_\alpha^\omega(x_y \cap \omega_1) = x_y \cap \alpha

for each \alpha \in x_y \setminus \omega_1. Then, because \text{s}up y = \text{s}up x_y, y = x_y for every y \in S_0. Hence S_0 \subseteq S \cap [\alpha^*]^\omega. Therefore S \cap [\alpha^*]^\omega is stationary. This completes the proof.

Now we turn our attention to Theorem 1.6. It is well-known that if a \lambda-supercardinal cardinal is Lévy collapsed to \omega_2 then SR([\lambda]^\omega) holds. It was shown by Shelah [10] that collapsing a \lambda-strongly compact cardinal suffices to obtain a model of SSR([\lambda]^\omega). First we review this:
Lemma 5.2 (Shelah [10] Ch.XIII, §1, 1.6.Claim, 1.10.Claim). Suppose that \( \kappa \) is a \( \lambda \)-strongly compact cardinal, where \( \lambda \) is a cardinal \( \geq \kappa \). Then \( \text{SSR}([\lambda]^\omega, < \kappa) \) holds. Moreover if \( \gamma \) is a regular uncountable cardinal < \( \kappa \) then \( \models_{\text{Col}(\gamma, \kappa)} \text{"SR}([\lambda]^\omega, < \kappa) \) “.

Proof. Both statements can be proved by similar arguments, but the latter is slightly harder than the former. We will prove only the latter.

We discuss some preliminaries in \( V \). Take a fine ultrafilter \( U \) over \( \mathcal{P}_\kappa \lambda \).

Let \( M \) be the transitive collapse of \( \text{Ult}(V, U) \), and let \( j : V \to M \) be the ultrapower map. Moreover, let \( f : \mathcal{P}_\kappa \lambda \to \mathcal{P}_\kappa \lambda \) be a function such that \( f(W) = W \cup \sup(W \cap \kappa) \) for each \( W \in \mathcal{P}_\kappa \lambda \) and let \( W^* := [f]_U \in M \). Then \( j^* \lambda \subseteq W^* \), and in \( M, W^* \in \mathcal{P}_{j(\kappa)} j(\lambda) \) and \( \omega_1 \subseteq W^* \cap j(\kappa) \in j(\kappa) \).

Suppose that \( \gamma \) is a regular uncountable cardinal < \( \kappa \) and that \( G \) is a \( \text{Col}(\gamma, < \kappa) \)-generic filter over \( V \). In \( V[G] \), take an arbitrary stationary \( S \subseteq [\lambda]^\omega \). We must show that, in \( V[G] \), there exists \( W \in \mathcal{P}_\kappa \lambda \) such that \( \omega_1 \subseteq W \cap \kappa \in \kappa \) and \( S \cap [W]^\omega \) is semistationary.

Let \( \bar{G} \) be a \( \text{Col}(\gamma, < j(\kappa)) \)-generic filter over \( V \) with \( \bar{G} \cap \text{Col}(\gamma, < \kappa) = G \). We work in \( V[\bar{G}] \). Define a map \( \bar{j} : V[\bar{G}] \to M[\bar{G}] \) by \( \bar{j}(\dot{a}) = j(\dot{a}) \) for each \( \text{Col}(\gamma, < \kappa) \)-name \( \dot{a} \in V \). Then \( \bar{j} \) is well-defined, and \( \bar{j} : V[\bar{G}] \to M[\bar{G}] \) is an elementary embedding which extends \( j \). For simplicity of notation, we let \( j \) denote \( \bar{j} \).

Note that \( S \) remains stationary because \( V[\bar{G}] \) is a \( \gamma \)-closed forcing extension of \( V[\bar{G}] \). Hence \( \{ j^* x \mid x \in S \} \) is stationary in \( [j^* \lambda]^\omega \). Moreover, for each \( x \in S \), \( j^* x = j(x) \) because \( x \) is countable in \( V[\bar{G}] \). Thus \( \{ j^* x \mid x \in S \} \subseteq j(S) \), and therefore \( j(S) \cap [j^* \lambda]^\omega \) is stationary. Then \( j(S) \cap [W^*]^\omega \) is semistationary by Lemma 2.3 (2). This is also true in \( M[\bar{G}] \). Hence \( W^* \) witnesses that the following holds in \( M[\bar{G}] \):

There exists \( W \in \mathcal{P}_{j(\kappa)} j(\lambda) \) such that \( \omega_1 \subseteq W \cap j(\kappa) \in j(\kappa) \) and \( j(S) \cap [W]^\omega \) is semistationary.

Therefore, by the elementarity of \( j \), it holds in \( V[\bar{G}] \) that there exists \( W \in \mathcal{P}_\kappa \lambda \) such that \( \omega_1 \subseteq W \cap \kappa \in \kappa \) and \( S \cap [W]^\omega \) is semistationary. This completes the proof.

We will prove that collapsing a \( \lambda \)-strongly compact cardinal does not suffice to obtain a model of \( \text{SR}([\lambda]^\omega) \). The core of Theorem 1.6 is the following theorem. As we see later, Theorem 1.6 will be obtained by further Lévy collapsing \( \kappa \) to \( 2 \).

Theorem 5.3. If \( \kappa \) is a supercompact cardinal, then there exists a generic extension in which \( \kappa \) is a strongly compact cardinal and \( \text{SR}([\kappa^+]^\omega, < \kappa) \) does not hold.

First we prove Theorem 5.3. Krueger [7] constructed a model in which \( \kappa \) is strongly compact but \( \text{S}(\kappa, \kappa^+) := \{ x \in \mathcal{P}_\kappa \kappa^+ \mid \text{o.t.}(x) = (x \cap \kappa)^+ \} \) is not stationary. (Note that \( \text{S}(\kappa, \kappa^+) \) is stationary if \( \kappa \) is \( \kappa^+ \)-supercompact.) We show that \( \text{SR}([\kappa^+]^\omega, < \kappa) \) does not hold in this model.
We start with a review of Krueger’s model. Krueger’s model was obtained from a model with $\kappa$ supercompact by two step forcing extension. The first step forces a partial square principle at $\kappa$ with preserving supercompactness of $\kappa$. This type of partial square principle was first introduced by Baumgartner in his unpublished note, and Apter-Cummings [1] showed that it can hold at a supercompact cardinal. In fact, the first step of Krueger’s construction is due to Apter-Cummings [1]. The second step is the iteration of Prikry forcing below $\kappa$ which was developed by Magidor [8]. We summarize basic properties of these forcings below.

**Definition 5.4.** For an uncountable cardinal $\kappa$ and an $E \subseteq \text{Lim}(\kappa^+)$, let $\square^E_\kappa$ be the following principle:

\[
\square^E_\kappa \equiv \text{There exists a sequence } \langle c_\beta \mid \beta \in E \rangle \text{ such that:} \\
(1) \ c_\beta \text{ is a club in } \beta. \\
(2) \ \text{if } \text{cf}(\beta) < \kappa \text{ then } \text{o.t.}(c_\beta) < \kappa, \\
(3) \ \text{if } \beta' \in \text{Lim}(c_\beta) \text{ then } c_{\beta'} = c_\beta \cap \beta', \\
\text{for each } \beta, \beta' \in E.
\]

We call a sequence $\langle c_\beta \mid \beta \in E \rangle$ satisfying (1)-(3) above a $\square^E_\kappa$-sequence.

The proof of the following fact can be also found in Krueger [7]:

**Fact 5.5 (Apter-Cummings [1]).** Assume that $\kappa$ is a supercompact cardinal. Then there exists a poset $\mathbb{P}$ with the following properties:

1. $\mathbb{P}$ preserves supercompactness of $\kappa$.
2. $\mathbb{P} \forces \square^E_\kappa$ holds for
   \[ E = \text{Lim}(\kappa^+) \setminus \bigcup \{ E^+_\alpha \mid \alpha \text{ is a measurable cardinal } < \kappa \} \].

**Fact 5.6 (Magidor [8]).** Assume that $\kappa$ is a supercompact cardinal. Then there exists a poset $\mathbb{Q}$ with the following properties:

1. $\mathbb{Q}$ has the $\kappa^+$-c.c.
2. $\mathbb{Q} \forces \text{"} \kappa \text{ is strongly compact".}$
3. For every measurable cardinal $\alpha < \kappa$, $\mathbb{Q} \forces \text{"} \text{cf}(\alpha) = \omega \text{".}$
4. For every measurable cardinal $\alpha < \kappa$, $\mathbb{Q}$ can be factored as $\mathbb{Q}_{\leq \alpha} \ast \mathbb{Q}_{> \alpha}$ so that $\mathbb{Q}_{\leq \alpha}$ has the $\alpha^+$-c.c. and $\mathbb{Q}_{> \alpha} \forces \text{"} \mathbb{Q}_{> \alpha} \text{ does not add subsets of } \alpha^+ \text{".}$

Before starting the proof of Theorem 5.3, we give a technical lemma:

**Lemma 5.7.** Suppose that $\kappa$ and $\theta$ are regular cardinals with $\omega_2 \leq \kappa < \theta$. Let $M$ be an elementary submodel of $\langle \mathcal{H}_\theta, \in \rangle$ such that $M \cap \kappa \in \kappa$ and such that both $\text{cf}(M \cap \kappa)$ and $\text{cf}(\text{sup}(M \cap \kappa^+))$ are uncountable. Then $M \cap \kappa^+$ is $\omega$-closed, that is, $\text{sup}(b) \in M$ for every countable $b \subseteq M \cap \kappa^+$.
Proof. We prove this by contradiction. Assume that \( b \subseteq M \cap \kappa^+ \) is countable and that \( \text{sup} \ b \notin M \cap \kappa^+ \). Then \( b \) does not have a greatest element. Also, \( b \) is bounded in \( M \cap \kappa^+ \) because \( \text{cf}(\text{sup}(M \cap \kappa^+)) = \text{uncountable} \). Let \( \beta^* \) be the least element of \((M \cap \kappa^+) \setminus \text{sup} b \). Note that \( \beta^* \) is a limit ordinal and \( \text{sup}(M \cap \beta^*) = \text{sup} \ b < \beta^* \).

Take an increasing continuous cofinal map \( \sigma : \text{cf}(\beta^*) \to \beta^* \).

First assume that \( \text{cf}(\beta^*) < \kappa \). Then \( \text{cf}(\beta^*) \subseteq M \) because \( \text{cf}(\beta^*) \in M \cap \kappa \in \kappa \). Hence \( \text{ran} \ \sigma \subseteq M \) and thus \( M \cap \beta^* \) is cofinal in \( \beta^* \). This contradicts \( \text{sup}(M \cap \beta^*) < \beta^* \).

Next assume that \( \text{cf}(\beta^*) = \kappa \). Then it is easy to see that \( \text{sup}(M \cap \beta^*) = \text{sup}(\text{supt}(M \cap \kappa)) \). Hence \( \text{cf}(\text{supt}(M \cap \beta^*)) = \text{uncountable} \) because \( \text{cf}(M \cap \kappa) \) is uncountable. But this contradicts \( \text{supt}(M \cap \beta^*) = \text{sup} \ b \) and \( b \) is countable.

This completes the proof. \( \square \)

Now we prove Theorem 5.3:

Proof of Theorem 5.3. Assume that \( \kappa \) is supercompact in \( V \). Let \( V_0 \) be a forcing extension of \( V \) by the poset \( P \) of Fact 5.5, and let \( V_1 \) be a forcing extension of \( V_0 \) by the poset \( Q \) of Fact 5.6. It suffices to show that \( \text{SR}([\kappa^+]^\omega, < \kappa) \) does not hold in \( V_1 \). Before starting, we summarize properties of \( V_0 \) and \( V_1 \).

In \( V_0 \), let \( E := \text{Lim}(\kappa^+) \setminus \bigcup\{E_{\alpha +}^\kappa : \alpha \text{ is a measurable cardinal } < \kappa \} \). Then \( \square^E_\kappa \) holds in \( V_0 \). Let \( \langle c_\gamma : \gamma \in E \rangle \) be a \( \square^E_\kappa \)-sequence. Note that, in \( V_0 \), there are unboundedly many measurable cardinals below \( \kappa \).

\( V_1 \) is a \( \kappa^+ \)-c.c. forcing extension of \( V_0 \). Moreover, in \( V_1 \), the following hold:

\((^1)\) If \( \alpha < \kappa \) is a measurable cardinal in \( V_0 \) the \( \text{cf} (\alpha) = \omega \).

\((^2)\) Suppose that \( \alpha < \kappa \) is measurable in \( V_0 \). Let \( \gamma := (\alpha^+)^{V_0} \). Then \( (E_\gamma^\kappa)^{V_0} \) is a stationary subset of \( E_\alpha^\kappa \).

\((^3)\) \( E_\alpha^\kappa \subseteq E \).

\((^2)\) and \((^3)\) hold by Fact 5.6 (4).

Now we show that \( \text{SR}([\kappa^+]^\omega, < \kappa) \) does not hold in \( V_1 \). We work in \( V_1 \). We show that there exists a stationary \( S \subseteq [\kappa^+]^\omega \) such that \( \{W \in \mathcal{P}_\kappa \kappa^+ : S \cap [W]^\omega \text{ is stationary} \} \) is nonstationary in \( \mathcal{P}_\kappa \kappa^+ \). By Lemma 2.8 this suffices. \( S \) will be constructed using Lemma 3.1.

First take a pairwise disjoint partition \( \langle A_\xi : \xi < \omega_1 \rangle \) of \( E_\omega^\kappa \) into stationary sets. Next take an injection \( \sigma : E_\omega^\kappa \to \kappa \) such that, for every \( \alpha \in \kappa \), \( \sigma(\alpha) > \alpha \) and \( \sigma(\alpha) \) is measurable in \( V_0 \). Then let \( B_\alpha := (E_{\sigma(\alpha)}^\kappa)^{V_0} \) for each \( \alpha \in E_\omega^\kappa \). Note the following:

- \( \langle B_\alpha : \alpha \in E_\omega^\kappa \rangle \) is a pairwise disjoint sequence of stationary subsets of \( E_\omega^\kappa \).
- For each \( \alpha \in E_\omega^\kappa \) and each \( \beta \in B_\alpha \), \( \text{cf} \beta^{V_0} > \alpha \).

Now let \( S \) be the set of all \( x \in [\kappa^+]^\omega \) such that

\(\begin{align*}
(1) \ s\text{up}(x \cap \kappa) & \in A_{\text{sup}(x \cap \omega_1)}, \\
(2) \ s\text{up}(x) & \in B_{\text{sup}(x \cap \kappa)}.
\end{align*}\)
The proof of this splits into two cases:

**Case 1:** $cf(M \cap \kappa) = \omega$ or $cf(\text{sup}(M \cap \kappa^+)) = \omega$.

First suppose that $cf(M \cap \kappa) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{sup}(x \cap \kappa) = M \cap \kappa\}$ is club in $[M \cap \kappa^+]^\omega$. Note that if $x, y \in S \cap C$ then $\text{sup}(x \cap \omega_1) = \text{sup}(y \cap \omega_1)$ by (1) of the construction of $S$. Thus $|\{\text{sup}(x \cap \omega_1) \mid x \in S \cap C\}| \leq 1$.

But $\omega_1 \subseteq M \cap \kappa^+$. Hence $S \cap C$ is nonstationary in $[M \cap \kappa^+]^\omega$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary.

The case when $cf(\text{sup}(M \cap \kappa^+)) = \omega$ is similar. Suppose that $cf(\text{sup}(M \cap \kappa^+)) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{sup}(x \cap \kappa) = \text{sup}(M \cap \kappa^+)\}$ is club in $[M \cap \kappa^+]^\omega$. If $x, y \in S \cap C$ then $\text{sup}(x \cap \kappa) = \text{sup}(y \cap \kappa)$ by (2) of the construction of $S$ and thus $\text{sup}(x \cap \omega_1) = \text{sup}(y \cap \omega_1)$ by (1). Hence $|\{x \cap \omega_1 \mid x \in S \cap C\}| \leq 1$.

Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. ■ (Case 1)

**Case 2:** Both $cf(M \cap \kappa)$ and $cf(\text{sup}(M \cap \kappa^+))$ are uncountable.

First we claim the following:

**Claim.** $cf^{V_\theta}(\text{sup}(M \cap \kappa^+)) \leq M \cap \kappa$.

Let $\delta := \text{sup}(M \cap \kappa^+)$. Then $\delta \in E$. Note that $M \cap \kappa^+$ is $\omega$-closed by Lemma 5.7. Hence, by (3), $\text{Lim}(c_\beta) \cap M \cap E$ is unbounded in $\delta$. Moreover if $\beta = \text{Lim}(c_\beta) \cap M \cap E$ then $a.t.(c_\beta \cap \beta) = a.t.(c_\beta) \in M \cap \kappa$. Thus $a.t.(c_\beta) \leq M \cap \kappa$. But $cf^{V_\theta}(\delta) \leq a.t.(c_\beta)$ because $c_\beta \in V_\theta$. Therefore $cf^{V_\theta}(\delta) \leq M \cap \kappa$.

Next suppose that $\delta \notin E$. Then there exists $\alpha < \kappa$ such that, in $V_\theta$, $\alpha$ is a measurable cardinal and $cf(\delta) = \alpha^+$. Then, by (4), $(E^{V_\theta}_\delta)$ is a stationary subset of $E^{\omega}_\delta$. On the other hand, $M \cap \kappa^+$ is $\omega$-closed unbounded in $\delta$. Thus there exists $\beta \in M \cap \kappa^+$ such that $cf^{V_\theta}(\beta) = \alpha$. Then, by the elementarity of $M$, $(\alpha^+)^{V_\theta} \in M \cap \kappa$. Therefore $cf^{V_\theta}(\delta) = (\alpha^+)^{V_\theta} \leq M \cap \kappa$.

This completes the proof of the claim. ■

Take an increasing continuous sequence $\langle \beta_\gamma \mid \gamma < cf^{V_\theta}(\text{sup}(M \cap \kappa^+)) \rangle \in V_\theta$ which is cofinal in $\text{sup}(M \cap \kappa^+)$. Let $C$ be the set of all $x \in [M \cap \kappa^+]^\omega$ such that, for some limit $\gamma < cf^{V_\theta}(\text{sup}(M \cap \kappa^+))$, $\text{sup}(x) = \beta_\gamma$ and $\gamma \leq \text{sup}(x \cap \kappa)$. Then $C$ is club in $[M \cap \kappa^+]^\omega$ by the claim above. Moreover if $x \in C$ and $\text{sup}(x) = \beta_\gamma$ then $\text{sup}(x \cap \kappa)$.

That is, $cf^{V_\theta}(\text{sup}(x \cap \kappa))$ for every $x \in C$. On the other hand, if $x \in S$ then $cf^{V_\theta}(\text{sup}(x \cap \kappa)) > \text{sup}(x \cap \kappa)$ by (2) of the construction of $S$. Thus $S \cap C = \emptyset$.

Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. ■ (Case 2)

This completes the proof of Theorem 5.3.
Now we turn our attention to Theorem 1.6. As we mentioned before, the forcing of Theorem 1.6 followed by Lévy collapsing $\kappa$ to $\omega_2$ gives Theorem 1.6.

Let $V_0, V_1$ and $E$ be as in the proof of Theorem 1.6. Let $V_2$ be an extension of $V_1$ by $\text{Col}(\omega_1, < \kappa)$. Then, by Lemma 5.2, SSR($[\lambda]^{\omega_1}$) holds in $V_2$ for every $\lambda \geq \omega_2$. So, by Lemma 2.5 (1) and 2.7 (1), it suffices to show that SR($[\omega_1]^{\omega_1}, < \omega_2$) does not hold in $V_2$.

Here note that $V_2$ is a $\kappa^{++}$-c.c. forcing extension of $V_0$ and that (*1), (*2) and (*3) all hold in $V_2$. (*2) holds because $\text{Col}(\omega_1, < \kappa)$ preserves stationary subsets of $E_2$. (*3) holds because $\text{Col}(\omega_1, < \kappa)$ preserves ordinals having uncountable cofinalities. Hence the same argument shows that SR($[\kappa^{++}]^{\omega_1}, < \kappa$) does not hold in $V_2$. But $\kappa = \omega_2$ in $V_2$. Therefore SR($[\omega_3]^{\omega_1}, < \omega_2$) does not hold in $V_2$.

References