Quantum system of set theory

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Key Words: quantum theory, logic, set theory
Subject code classifications: 03, 06, 81

Abstract

In this paper we formulate the quantale valued set theory and interprete quantum theory in the system $(V^{P(\mathcal{H})},\ V^{\mathcal{Q}})$ of quantum universe $V^{P(\mathcal{H})}$ and quantale valued universe $V^{\mathcal{Q}}$.

The truth value set of quantum theory may be considered as being the complete orthomodular lattice $P(\mathcal{H})$ of all projections on a Hilbert space $\mathcal{H}$. $P(\mathcal{H})$ has the structure of the sheaf $\mathrm{Sh}_{\mathcal{U}}(B)$ of the complete Boolean lattices over the topological group $\mathcal{U}$ of unitary operators on $\mathcal{H}$. The sheaf $\mathrm{Sh}_{\mathcal{U}}(B)$ is extended to the sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$ of the copies of the Boolean valued universe V^B over $\mathcal{U}$. A state of the physical system is represented by a cross-section that is a 'complex number' in each stalk V^{B_U}, and a quantum proposition is represented by a multiplication operator by 'real number' which is equal to either 1 or 0 in some stalk.

$P(\mathcal{H})$ can be further extended to the quantale $\mathcal{Q} = Q(P(\mathcal{H}))$ of endomorphisms of $P(\mathcal{H})$. Quantale valued set theory with the logical connective & ('and then') is developed in the $\mathcal{Q}$-valued universe $V^{\mathcal{Q}}$. The quantum universe $V^{P(\mathcal{H})}$ can be embedded in $V^{\mathcal{Q}}$, as both left-universe and right-universe.

We show that the quantum theory in the classical mathematical framework can be translated to a theory about properties of complex numbers in our quantum system of universes. Using this kind of translation, we interprete e.g. 'double slit experiment' and 'entangled state of spins' in our quantum system of set theory, where, we believe, their seeming paradoxes appearing in the classical framework are well resolved.

Contents

1 Introduction

In the general mathematical formulation of quantum physics, states of a physical system are represented by vectors in a Hilbert space, and observables are represented by self-adjoint operators. Let $\mathcal{H}$ be a Hilbert space representing a physical system. When an observable is represented by a self-adjoint operator, a state of the physical system represented by an eigen vector of the self-adjoint operator is called an **eigen-state** of the observable.

In the quantum theory, an observable which is 'either 1(true) or 0(false)' in their eigen-states is considered as proposition, and called a **quantum proposition**. That is, quantum propositions are represented by projections.

The set $P(\mathcal{H})$ of all projections on $\mathcal{H}$ is a complete orthomodular lattice with respect to the order of inclusion of their ranges. $P(\mathcal{H})$, which represents the set of quantum propositions, is considered as the truth value set of quantum theory. The structure of the truth value set is represented by a logic. The mathematical language of lattice was first introduced into quantum physics by Birkhoff and von Neumann [1].

Let V be a universe of ZFC. The universe $V^{P(\mathcal{H})}$ with truth value set $P(\mathcal{H})$ is constructed in V by transfinite induction as follows:

$$\begin{aligned} V_\alpha^{P(\mathcal{H})} &= \{u \mid \exists \beta < \alpha\, \exists \mathcal{D}u \subset V_\beta^{P(\mathcal{H})} (u : \mathcal{D}u \to P(\mathcal{H}))\} \\ V^{P(\mathcal{H})} &= \textstyle\bigcup_{\alpha \in \mathrm{On}} V_\alpha^{P(\mathcal{H})}. \end{aligned}$$

In [10] and [11], G.Takeuti developed a quantum set theory in the universe $V^{P(\mathcal{H})}$, and showed that 'real numbers' defined in the quantum set theory are self-adjoint operators on $\mathcal{H}$ in the external universe V. That is, an observable is represented by a real number in $V^{P(\mathcal{H})}$.

In [5], M.Ozawa extended the Takeuti's formulation to construct a model of set theory, which is based on the logic represented by the lattice of projections in an arbitrary von Neumann algebra, and established transfer principle of ZFC to their quantum counterparts holding in the model.

Let $\{e_j\}_{j\in J}$ be an arbitrary but fixed basis of $\mathcal{H}$. Each e_j corresponds to the projection p_j with range $\mathbb{C}e_j$, which is an atom in $P(\mathcal{H})$. Let

$$P = \{p_j\}_{j\in J} \quad \text{and} \quad B = \{\bigvee_{j\in K} p_j \mid K \subset J\}.$$

P is a **complete orthogonal system**, that is, a set of mutually orthogonal atoms such that $\bigvee_{j\in J} p_j = 1$. B is a sublattice of $P(\mathcal{H})$ generated by P, which is a complete Boolean lattice isomorphic to the power set $\mathcal{P}(J)$ of J.

Let $\mathcal{U}$ be the topological group of all unitary operators acting on $\mathcal{H}$, and let

$$\sigma_U(p) = UpU^* \quad \text{for} \quad p \in P(\mathcal{H}) \text{ and } U \in \mathcal{U}.$$

σ_U is an automorphism of $P(\mathcal{H})$. For each $U \in \mathcal{U}$, let

$$P_U = \{\sigma_U(p_j) \mid j \in J\} \quad \text{and} \quad B_U = \{\bigvee_{j\in K} \sigma_U(p_j) \mid K \subset J\}.$$

P_U is a complete orthogonal system, and B_U is a complete Boolean sublattice of $P(\mathcal{H})$ generated by P_U.

$P(\mathcal{H})$ is the union $\bigcup_{U\in\mathcal{U}} B_U$ of mutually isomorphic complete Boolean sublattices of $P(\mathcal{H})$.

$$P(\mathcal{H}) = \bigcup_{U\in\mathcal{U}} B_U \qquad \text{(Theorem 2.13)}$$

It follows that $P(\mathcal{H})$ has the structure of the sheaf of complete Boolean lattices over $\mathcal{U}$:

$$\mathrm{Sh}_{\mathcal{U}}(B) = < \widetilde{B}, \pi >, \quad \text{where}$$

$$\begin{array}{rcl} \widetilde{B} & = & \{< U, \sigma_U(a) > \mid U \in \mathcal{U},\ a \in B\}, \\ \pi & : & \widetilde{B} \to \mathcal{U}, \quad \text{where } \pi :< U, \sigma_U(a) > \mapsto U \\ \mathcal{O}(\mathcal{U}) & \text{is} & \text{the set of open sets of } \mathcal{U} \\ \Gamma(X, a) & = & \{< U, \sigma_U(a) > \mid U \in X\} \quad \text{for } X \in \mathcal{O}(\mathcal{U}),\ a \in B \end{array}$$

$\widetilde{B}$ is a topological space with the open base $\{\Gamma(X, a) \mid X \in \mathcal{O}(\mathcal{U}),\ a \in B\}$, and π is a locally homeomorphism.

Boolean valued universe V^{B_U} is a sub-universe of $V^{P(\mathcal{H})}$ for each $U \in \mathcal{U}$.

$$\bigcup_{U\in\mathcal{U}} V^{B_U} \subset V^{P(\mathcal{H})}$$

$\sigma_U : B \to B_U$ is extended to an isomorphism $\sigma_U : V^B \to V^{B_U}$ and $\{V^{B_U} \mid U \in \mathcal{U}\}$ forms a sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$ of Boolean valued universes over $\mathcal{U}$:

$$\mathrm{Sh}_{\mathcal{U}}(V^B) = < \widetilde{V^B}, \pi >, \quad \text{where}$$

$$\widetilde{V^B} = \{< U, x > \mid U \in \mathcal{U},\ x \in V^{B_U}\}, \quad \pi :< U, x > \mapsto U,$$

$$\Gamma(X, x) = \{< U, \sigma_U(x) > \mid U \in X, \} \quad \text{for a open set } X \text{ of } \mathcal{U} \text{ and } x \in V^B.$$

A cross-section of the sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$ which is a complex number in each stalk V^{B_U} is called a complex number in the sheaf. Each state of the quantum system, which is represented by a vector of $\mathcal{H}$, is represented by a complex number in the sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$. The set of complex numbers in the sheaf may be considered as the state space of the quantum system. Each observable, which is represented by a self-adjoint operator, is an operator on the state space, that is represented by a multiplication operator by 'real number' in some stalk.

The logic in the universe $V^{P(\mathcal{H})}$ represents the structure $< P(\mathcal{H}), \bigwedge, \bigvee, ^{\perp} >$ of the truth value set. In order to develop a set theory in $V^{P(\mathcal{H})}$, we need a logical implication. G.Takeuti introduced an implication defined as

$$\varphi \to \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^{\perp} \vee (\varphi \wedge \psi).$$

This implication is considered as an expression of a local lattice order. The global lattice order $\leq$ is not an operator but a relation on $P(\mathcal{H})$. Hence, $\leq$ represents a meta-logical notion. That is, $a \leq b$ represents a proposition in the external universe V with the truth value set $\mathbf{2}(= \{1,0\})$. Since $\mathbf{2} \subset P(\mathcal{H})$, the lattice order $\leq$ is expressed by operator $\to_{\mathbf{2}}$: $P(\mathcal{H}) \times P(\mathcal{H}) \to \mathbf{2}$ defined by $a \to_{\mathbf{2}} b = \begin{cases} 1 & a \leq b \\ 0 & a \not\leq b \end{cases}$.

In [12], Titani formulated a lattice valued logic L and a lattice valued set theory LZFZ (Lattice valued Zermelo-Faenkel set theory with Zorn's lemma) by introducing the logical connective $\to_{\mathbf{2}}$ representing the order relation $\leq$. $\to_{\mathbf{2}}$ is called **global (or basic) implication**. Then in [13], she formulated a **quantum logic** QL representing complete orthomodular lattice, and a **quantum set theory** QZFZ (Quantum Zermelo-Faenkel set theory with Zorn's lemma) based on the logic QL. $V^{P(\mathcal{H})}$ is a model of QZFZ.

By virtue of the logical operation $\to_{\mathbf{2}}$, the external universe V which is represented by $V^{\mathbf{2}}$ $(V \cong V^{\mathbf{2}} \subset V^{P(\mathcal{H})} \subset V)$, can be described in terms of language of QZFZ. The truth value set $(\cong \mathcal{P}(1))$ is also described in terms of language of QZFZ. Using this fact, a completeness of QZFZ in the following sense was proved :
For a sentence φ of QZFZ,

$$\begin{aligned} \text{ZFC} \quad \vdash \quad & \text{“ } \varphi \text{ is valid in } V^{Q} \text{ for all complete orthomodular} \\ & \text{lattice } Q \text{ ”} \\ \Longrightarrow \quad & \text{QZFZ} \vdash \varphi. \end{aligned}$$

In Section 3, $P(\mathcal{H})$ is extended to a quantale $Q(P(\mathcal{H}))$ which is a complete lattice consisting of endomorphisms of $P(\mathcal{H})$. Quantale is a complete lattice with an operation & representing the composition of endomorphisms, and with an involution (Definition 3.5). The quantale was introduced in C.J.Mulvey and J.Wick Pelletier [3] and P.Resende [7]. Let $\mathcal{Q} = Q(P(\mathcal{H}))$. $P(\mathcal{H})$ is embedded in $\mathcal{Q}$ as both right subquantale $R(\mathcal{Q})$ and left subquantale $L(\mathcal{Q})$.

$$\chi_R : P(\mathcal{H}) \cong R(\mathcal{Q}) \subset \mathcal{Q}, \quad \chi_L : P(\mathcal{H}) \cong L(\mathcal{Q}) \subset \mathcal{Q}.$$

The embeddings are extended to the embeddings of universes:

$$\chi_R : V^{P(\mathcal{H})} \cong V^{R(\mathcal{Q})} \subset V^{\mathcal{Q}}, \quad \chi_L : V^{P(\mathcal{H})} \cong V^{L(\mathcal{Q})} \subset V^{\mathcal{Q}}.$$

We formulate a quantale valued set theory QtZFZ (Quantale valued Zermelo-Faenkel set theory with Zorn's lemma) which is developed in the quantale valued universe $V^{\mathcal{Q}}$. QtZFZ is a lattice valued set theory provided with a propositional constant $\mathfrak{e}$ and logical connectives &, $^{\perp}$, $*$ together with additional axioms for them. QtZFZ is complete in the following sense: For a sentence φ of QtZFZ,

$$\begin{aligned} \text{ZFC} \quad \vdash \quad & \text{“ } \varphi \text{ is valid in } V^{Q} \text{ for all Hilbert quantale } Q \text{ (Definition 3.15) ”} \\ \Longrightarrow \quad & \text{QtZFZ} \vdash \varphi. \end{aligned}$$

A vector x in $\mathcal{H}$ has the expression $\sum_{j\in J}(Ue_i, x)\cdot Ue_j$ for each $U\in\mathcal{U}$, where $(\ ,\ )$ is the inner product. Since vector Ue_j is represented in V^{B_U} by a real number $\widehat{\sigma_U(p_j)}$ corresponding to the projection $\sigma_U(p_j)$, the expression $\sum_{j\in J}(Ue_i, x)\cdot Ue_j$ is represented by a complex number in V^{B_U} :

$$\widehat{x}_U = \sum_{j\in J}(Ue_j, x)\check{}\cdot\widehat{\sigma_U(p_j)} \in V^{B_U}\subset V^{P(\mathcal{H})}.$$

Accordingly, $x\in\mathcal{H}$ is represented by the cross-section $\{< U, \widehat{x}_U >\mid U\in\mathcal{U}\}$ of the sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$.

Coordinates of $\widehat{x}_U$ with respect to $\{\widehat{\sigma_U(p_j)}\mid j\in J\}$ is a check set, i.e. $\in V^{\mathbf{2}}$. Since $V^{\mathbf{2}}$ represents the external universe V, the coordinate matrix $[(Ue_j, x)\check{}]_{j\in J}$ of $\widehat{x}_U$ with respect to $\{\widehat{\sigma_U(p_j)}\mid j\in J\}$ can be considered as the coordinate matrix $[(Ue_j, x)]_{j\in J}$ in V, which is denoted by $[\widehat{x}_U]$.

$$[\widehat{x}_U] = [(Ue_j, x)]_{j\in J} = \begin{bmatrix} \vdots \\ (Ue_j, x) \\ \vdots \end{bmatrix} = [U]\cdot[\widehat{x}_E]$$

where $[U] = \begin{bmatrix} & \vdots & \\ \cdots & (Ue_i, e_j) & \cdots \\ & \vdots & \end{bmatrix} = [(Ue_i, e_j)]_{ij}$ and $[\widehat{x}_E] = \begin{bmatrix} \vdots \\ (e_j, x) \\ \vdots \end{bmatrix}$.

Let

$$\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}) = < \widetilde{\mathcal{H}}, \pi >, \quad \text{where}$$

$$\widetilde{\mathcal{H}} = \{< U, [\widehat{x}_U] >\mid U\in\mathcal{U},\ x\in\mathcal{H}\} \in V, \quad \text{and} \quad \pi(< U, [\widehat{x}_U] >) = U.$$

$\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ is a sheaf of Hilbert spaces over $\mathcal{U}$, and $x\in\mathcal{H}$ is represented by the cross-section $\{< U, [\widehat{x}_U] >\mid U\in\mathcal{U}\}$ of the sheaf.

Since $V^{P(\mathcal{H})}$ is embedded in $V^{\mathcal{Q}}$ by χ_R and χ_L, each vector $x\in\mathcal{H}$ is represented in $V^{\mathcal{Q}}$ as the cross-section $\chi_R(\widehat{x}) = \{< U, \chi_R(\widehat{x}_U) >\mid U\in\mathcal{U}\}$ of $\chi_R(\mathrm{Sh}_{\mathcal{U}}(V^B))$ in $V^{R(\mathcal{Q})}$, and similarly, the cross-section $\chi_L(\widehat{x}) = \{< U, \chi_L(\widehat{x}_U) >\mid U\in\mathcal{U}\}$ in $V^{L(\mathcal{Q})}$.

Each $\chi_R(\widehat{x}_U)$ is represented by the matrix $\{(Ue_j, x)\}_{j\in J}$ in V, with which the matrix $\{(Ue_j, x)\}\check{}_{j\in J}$ with respect to the basis $\{\chi_R(\widehat{\sigma_U(p_j)})\}_{j\in J}$ is associated. The matrix $\{(Ue_j, x)\}_{j\in J}$ representing $\chi_R(\widehat{x}_U)$ is denoted by $|x\rangle_U$, and the matrix $\{\overline{(Ue_j, x)}\}_{j\in J}$ representing $\chi_L(\widehat{x}_U)$ is denoted by $\langle x|_U$.

$$|x\rangle_U = \overline{\langle x|}_U = [(Ue_j, x)]_{j\in J} = \begin{bmatrix} \vdots \\ (Ue_j, x) \\ \vdots \end{bmatrix} = [U]\cdot|x\rangle_E.$$

$\bigcup_{U\in\mathcal{U}}\{|x\rangle_U \mid x\in\mathcal{H}\}$ forms the sheaf $\mathrm{Sh}^{R}_{\mathcal{U}}(\mathcal{H})$ of Hilbert spaces:

$$\mathrm{Sh}^{R}_{\mathcal{U}}(\mathcal{H}) =< \widetilde{\mathcal{H}}^R, \pi >, \quad \text{where}$$

$$\widetilde{\mathcal{H}}^R = \{< U, |x\rangle_U > \mid U\in\mathcal{U},\ x\in\mathcal{H}\}, \quad \pi :< U, |x\rangle_U > \mapsto U.$$

The state of the quantum system which is represented by $x\in\mathcal{H}$ is represented by the cross-section $|x\rangle$ of the sheaf.

$$|x\rangle = \{< U, |x\rangle_U > \mid U\in\mathcal{U}\}.$$

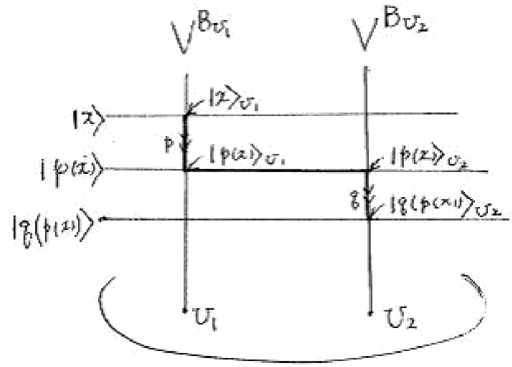
$\mathrm{Sh}^{R}_{\mathcal{U}}(\mathcal{H})$ and $\mathrm{Sh}^{L}_{\mathcal{U}}(\mathcal{H})$ are both considered as the state space.

An observable, which is represented by a self-adjoint operator on $\mathcal{H}$, is represented by an operator on the state spaces $\mathrm{Sh}^{R}_{\mathcal{U}}(\mathcal{H})$ and $\mathrm{Sh}^{L}_{\mathcal{U}}(\mathcal{H})$ (Theorem 3.16). Especially, the proposition represented by $p\in P(\mathcal{H})$ is represented by a real number in some $V^{B_U} \subset V^{P(\mathcal{H})}$, and $\widehat{p}\cdot\widehat{x}_U = \widehat{p(x)}_U$. Thus, the proposition represented by $p\in P(\mathcal{H})$ is represented by an operator $|x\rangle \mapsto |p(x)\rangle$ on $\mathrm{Sh}^{R}_{\mathcal{U}}(\mathcal{H})$. We denote the operator by $|p\rangle\langle p|$ using the Dirac's notation:

$$|p\rangle\langle p| : |x\rangle \mapsto |p(x)\rangle \quad \text{for } x\in\mathcal{H}$$

If φ, ψ are propositions represented by $p_\varphi\in B_{U_1}$ and $p_\psi\in B_{U_2}$, then "φ and then ψ", in symbols $\varphi\&\psi$, is represented by the product of propositions. That is, first $|p_\varphi(x)\rangle_{U_1}$ is measured in $V^{B_{U_1}}$ and then $|p_\psi(p_\varphi(x))\rangle_{U_2}$ is measured in $V^{B_{U_2}}$.

$$\Big(|p_\varphi\rangle\langle p_\varphi| \,\&\, |p_\psi\rangle\langle p_\psi|\Big)(|x\rangle) = |p_\psi(p_\varphi(x))\rangle$$



In Section 6, we interprete the rules of quantum theory, 'Double slit experiment' and 'entangled state of spins' in $V^{\mathcal{Q}}$, and obtain the interpretation in quantum theory.

The author wishes to express the deepest appreciation to Professor Gaisi Takeuti for leading her to the study of quantum set theory, and for his support and encouragement ever since 1980. She also wishes to express her gratitude to Professors Sakae Fuchino and Heiji Kodera for their great interest to this project and for helpful discussions and valuable advices.

2 Quantum set theory

2.1 Proposition system $P(\mathcal{H})$

A quantum proposition is represented as a projection acting on a Hilbert space. Let $\mathcal{H}$ be a Hilbert space. Projections on $\mathcal{H}$ form a complete orthomodular lattice with respect to the order of inclusion of their ranges, which is denoted by $P(\mathcal{H})$:

$$P(\mathcal{H}) \stackrel{\text{def}}{=} \{p : \mathcal{H} \to \mathcal{H} \mid p \text{ is a projection on } \mathcal{H}\},$$

$$p \leq q \stackrel{\text{def}}{\Longleftrightarrow} M_p \subset M_q, \qquad M_{p^\perp} \stackrel{\text{def}}{=} (M_p)^\perp, \quad \text{for } p, q \in P(\mathcal{H}),$$

$$\text{where } M_p = \{p(x) \mid x \in \mathcal{H}\}.$$

Definition 2.1. A lattice $\mathcal{L}$ is said to be **orthocomplemented** if it is provided with an operator $^\perp$ satisfying

(C) (C_1) $c^{\perp\perp} = c$

(C_2) $c \vee c^\perp = 1$, and $c \wedge c^\perp = 0$

(C_3) $b \leq c \Longrightarrow c^\perp \leq b^\perp$ for $\forall\, b \in \mathcal{L}$

Definition 2.2. A sublattice of a complete orthocomplemented lattice $\mathcal{L}$ which is closed under $\bigvee$, $^\perp$ is also a complete orthocomplemented lattice, and called a $(\bigvee, {}^\perp)$**-sublattice** of $\mathcal{L}$. A complete orthocomplemented lattice $\mathcal{L}$ is said to be **Boolean**, if it is distributive.

Definition 2.3. An orthocomplemented lattice $\mathcal{L}$ is **orthomodular** if the following condition (**P**) is satisfied :

(P) If $b, c \in \mathcal{L}$ and $b \leq c$, then the sublattice of $\mathcal{L}$ generated by $\{b, b^\perp, c, c^\perp\}$ is distributive, i.e. Boolean $(\bigvee, {}^\perp)$-sublattice.

Definition 2.4. In a orthomodular lattice $\mathcal{L}$, one say that c **covers** b when

1. $b < c$, and
2. if $b \leq x \leq c$ then $x = b$ or $x = c$.

An element which covers 0 is called an **atom**.

Definition 2.5 (Piron [6])**.** A complete orthomodular lattice $\mathcal{L}$ satisfying the following condition (**A**) is called a **proposition system**.

(A) (A_1) If $b \in \mathcal{L}$ is different from 0, there exists an atom $p \leq b$.

(A_2) If p is an atom and if $p \wedge b = 0$, then $p \vee b$ covers b.

Theorem 2.6. *$P(\mathcal{H})$ is a proposition system.*

Definition 2.7. Elements b, c of a complete orthomodular lattice $\mathcal{L}$ are said to be **compatible**, in symbols $c \underset{\circ}{\mid} b$, if the sublattice generated by $\{b, b^\perp, c, c^\perp\}$ is distributive.

If $b \in \mathcal{L}$ and $A \subset \mathcal{L}$,

$$b \underset{\circ}{\mid} A \overset{\text{def}}{\iff} \forall a \in A\, (b \underset{\circ}{\mid} a), \qquad \underset{\circ}{\mid} A \overset{\text{def}}{\iff} \forall a, b \in A\, (a \underset{\circ}{\mid} b).$$

Theorem 2.8 (Piron [6])**.** *If $\mathcal{L}$ is a complete orthomodular lattice, then*

1. $b \underset{\circ}{\mid} c \iff c = (b \wedge c) \vee (b^\perp \wedge c)$ *for $b, c \in \mathcal{L}$.*
2. $b \underset{\circ}{\mid} c \iff c \underset{\circ}{\mid} b \iff b^\perp \underset{\circ}{\mid} c$ *for $b, c \in \mathcal{L}$.*

Theorem 2.9 (Piron [6])**.** *Let $\mathcal{L}$ be a complete orthomodular lattice. If $b \in \mathcal{L}$, $C \subset \mathcal{L}$ and $b \underset{\circ}{\mid} C$, then*

$$\bigvee_{c \in C} (b \wedge c) = b \wedge (\bigvee C), \quad \bigwedge_{c \in C} (b \vee c) = b \vee (\bigwedge C),$$

$$\textit{and hence,} \qquad b \underset{\circ}{\mid} \bigvee C, \quad b \underset{\circ}{\mid} \bigwedge C.$$

Theorem 2.10. *If a subset C of a complete orthomodular lattice $\mathcal{L}$ is compatible, then there exists a Boolean $(\bigvee, {}^\perp)$-sublattice of $\mathcal{L}$ including C.*

Proof. Let $X' \overset{\text{def}}{=} \{a \in \mathcal{L} \mid a \underset{\circ}{\mid} b \text{ for all } b \in X\}$ for $X \subset \mathcal{L}$, and assume $\underset{\circ}{\mid} C$. Then $C \subset C'$, hence $C'' \subset C'$. $c \in C$ implies $c \underset{\circ}{\mid} b$ for all $b \in C'$. It follows that $C \subset C'' \subset C'$.

By Theorem 2.9, C'' is a Boolean $(\bigvee, {}^\perp)$- sublattice. ☐

Definition 2.11. A set $\{q_j\}_{j \in J}$ of mutually orthogonal atoms in a proposition system $\mathcal{L}$ is called a **complete orthogonal system** if $\bigvee_{j \in J} q_j = 1$.

Notations 2.12. Throughout this paper we use the following notations.

- $\mathcal{H}$ is a Hilbert space.
- $P(\mathcal{H})$ is the proposition system consisting of all projection
- $\{e_j\}_{j \in J}$ is a fixed basis of $\mathcal{H}$.
- p_j is the projection whose range is the span of singleton $\{e_j\}$, i.e. p_j is an atom (Definition 2.7) of $P(\mathcal{H})$ such that $p_j(e_j) = e_j$.
- $P \overset{\text{def}}{=} \{p_j\}_{j \in J}$. P is a complete orthogonal system in $P(\mathcal{H})$ (Definition 2.11).
- $B \overset{\text{def}}{=} \{\bigvee_{j \in K} p_j \mid K \subset J\}$. B is a Boolean $(\bigvee, {}^\perp)$-sublattice of $P(\mathcal{H})$, which is isomorphic to $\mathcal{P}(J)$.

- $\mathcal{U}$ is the topological group consisting of all unitary operators on $\mathcal{H}$, where identity operator is denoted by E. Elements of $\mathcal{U}$ are denoted by $U, U', \cdots$. $\mathcal{O}(\mathcal{U})$ is the set of open sets of $\mathcal{U}$.

- $\sigma_U(p) \overset{\text{def}}{=} UpU^* \quad$ for $p \in P(\mathcal{H})$ and $U \in \mathcal{U}$.
  σ_U is an automorphism on $P(\mathcal{H})$ preserving $\bigvee$ and $^\perp$.

- $P_U \overset{\text{def}}{=} \{\sigma_U(p_j)\}_{j\in J}, \;\; B_U \overset{\text{def}}{=} \{\bigvee_{j\in K} \sigma_U(p_j) \mid K \subset J\}$ for each $U \in \mathcal{U}$.
  P_U is a complete orthogonal system in $P(\mathcal{H})$, and B_U is a Boolean $(\bigvee, ^\perp)$-sublattice of $P(\mathcal{H})$, which is isomorphic to $\mathcal{P}(J)$.

For any complete orthogonal system $\{q_j\}_{i\in J}$ of $P(\mathcal{H})$, there exists a unitary operator $U \in \mathcal{U}$, such that

$$q_j = \sigma_U(p_j) \quad \text{for each } j \in J,$$

and for each $q \in P(\mathcal{H})$, there exists a unitary operator $U \in \mathcal{U}$ and $p \in B$ such that $q = \sigma_U(p) = \bigvee\{\sigma_U(p_j) \mid \sigma_U(p_j) \leq q\}$. Hence,

Theorem 2.13. *$P(\mathcal{H})$ is the union of mutually isomorphic Boolean $(\bigvee, ^\perp)$-sublattices of $P(\mathcal{H})$.*

$$P(\mathcal{H}) = \bigcup_{U\in\mathcal{U}} B_U, \quad \textit{where} \quad B_U = \{\sigma_U(p) \mid p \in B\}.$$

Definition 2.14. Let

$$\widetilde{B} = \{< U, \sigma_U(p) > \mid \; U \in \mathcal{U}, \; p \in B\}, \quad \text{and}$$

$$\Gamma(X, p) = \{< U, \sigma_U(p) > \mid \; U \in X\} \quad \text{for } p \in B, \; X \in \mathcal{O}(\mathcal{U})$$

$$\pi(< U, \sigma_U(p) >) = U \quad \text{for} \quad < U, \sigma_U(p) > \in \mathfrak{B}.$$

$\widetilde{B}$ is a topological space with open base $\{\Gamma(X, p) \mid X \in \mathcal{O}(\mathcal{U}), \; p \in B\}$, and $\pi : \widetilde{B} \to \mathcal{U}$ is a homeomorphism. Then the pair $< \widetilde{B}, \pi >$ is a **sheaf of complete Boolean lattice over** $\mathcal{U}$, and is denoted by $\text{Sh}_{\mathcal{U}}(B)$.

$$\text{Sh}_{\mathcal{U}}(B) =< \widetilde{B}, \pi > .$$

$\Gamma(X, B) = \{\Gamma(X, p) \mid p \in B\}$ is a complete Boolean lattice for each $X \in \mathcal{O}(\mathcal{U})$, where $\bigvee_\alpha \Gamma(X, p_\alpha) = \Gamma(X, \bigvee_\alpha p_\alpha)$ and $\Gamma(X, p)^\perp = \Gamma(X, p^\perp)$.

2.2 Implications on a complete lattice

Let $\mathcal{L}$ be a complete lattice. The top $\bigvee \mathcal{L}$ is denoted by 1 and the bottom $\bigwedge \mathcal{L}$ is denoted by 0.

A binary operation $\to$ on a lattice $\mathcal{L}$ is called an **implication** if

1. $(a \to b) = 1 \iff a \leq b,$ and
2. $a \wedge (a \to b) \leq b.$

A complete lattice with implication is considered as a representation of some logic.

Implication $\to$ is said to be **transitive** if

$$(a \to b) \wedge (b \to c) \leq (a \to c) \quad \text{for } a, b, c \in \mathcal{L}.$$

Transitivity of the corresponding logical implication is indispensable for the development of set theory, since equality axioms of set theory which depends on the transitivity of implication is fundamental.

Algebraic operations on a lattice is represented by logical operations. While the order relation of the lattice is a 'relation' rather than operation. Relation on a lattice is represented by a meta-logical notion. We suppose our meta-logic is a classical logic. Then, the truth value of $a \to b$ should be in a Boolean lattice representing the meta-logic.

Definition 2.15. A complete sublattice Z of a complete lattice $\mathcal{L}$ is said to be a **center** of $\mathcal{L}$ if

$$A \subset Z \text{ and } C \subset \mathcal{L} \implies \bigvee A \wedge \bigvee C = \bigvee_{a \in A,\, c \in C} a \wedge c.$$

Definition 2.16. If $\mathcal{L}$ is a complete lattice with a center Z, then we define operators $\to_Z$, $\neg_Z$, $\Box_Z$, $\Diamond_Z$ by

$$(a \to_Z b) \stackrel{\text{def}}{=} \bigvee \{c \in Z \mid a \wedge c \leq b\}, \quad \neg_Z\, a \stackrel{\text{def}}{=} (a \to_Z 0),$$

$$\Box_Z\, a \stackrel{\text{def}}{=} \bigvee \{x \in Z \mid x \leq a\}, \quad \Diamond_Z\, a \stackrel{\text{def}}{=} \bigwedge \{x \in Z \mid a \leq x\}.$$

Theorem 2.17. *If $\mathcal{L}$ is a complete lattice with a center Z, then*

1. *$\to_Z$ is a transitive implication on $\mathcal{L}$.*
2. *$a \wedge c \leq b \iff c \leq (a \to_Z b)$ for $a, b, c \in \mathcal{L}$.*
3. *$(a \to_Z b), (\Box_Z a), (\neg_Z a), (\Diamond_Z a) \in Z$.*
4. *$a \in Z \iff a = \Box_Z a = \Diamond_Z a$.*

5. *If* $a \in Z$*, then* $\neg_Z a$ *is the Boolean complement of* a *in* Z*, i.e.*

$$a \vee \neg_Z a = 1 \quad \text{and} \quad a \wedge \neg_Z a = 0.$$

6. *If* Z, Z' *are centers of* $\mathcal{L}$ *and* $Z' \subset Z$*, then* $(a \rightarrow_{Z'} b) \leq (a \rightarrow_Z b)$ *for* $a, b \in \mathcal{L}$*.*

Proof. Since 2., 3., 4., 6. are obvious, we prove 1. and 5.

$$\begin{aligned} 1.\ (a \rightarrow_Z b) \wedge (b \rightarrow_Z c) &= \bigvee\{x \in Z \mid a \wedge x \leq b\} \wedge \bigvee\{y \in Z \mid b \wedge y \leq c\} \\ &= \bigvee\{x \wedge y \mid x \in Z,\ y \in Z,\ a \wedge x \leq b,\ b \wedge y \leq c\} \\ &\leq \bigvee\{z \in Z \mid a \wedge z \leq c\} = (a \rightarrow_Z c). \end{aligned}$$

5. For $a \in Z$, let $b \in Z$ be the Boolean complement of a, that is, $a \vee b = 1$ and $a \wedge b = 0$. Then $b \leq \neg_Z a$ by $a \wedge b \leq 0$; and $\neg_Z a \leq b$, since $\neg_Z a = \neg_Z a \wedge (a \vee b) = (\neg_Z a \wedge a) \vee (\neg_Z a \wedge b) \leq b$. Therefore, $b = \neg_Z a$.

□

2 ($=< \{1, 0\}, \leq >$) is a center for every complete lattice. Especially, **2** is the only center of $P(\mathcal{H})$.

Theorem 2.18. *Let* $\mathcal{L}$ *be a complete lattice. If* $a, b, c \in \mathcal{L}$ *and* $\{a_i\}_i$*,* $\{b_i\}_i \subset \mathcal{L}$*, then:*

1. $\Box_{\mathbf{2}} a = (1 \rightarrow_{\mathbf{2}} a) \in \mathbf{2}$; $\quad \neg_{\mathbf{2}} a = (a \rightarrow_{\mathbf{2}} 0) \in \mathbf{2}$.
2. *If* $a, b \in \mathbf{2}$*, then* $(a \rightarrow_{\mathbf{2}} b) = \neg_{\mathbf{2}} a \vee b$.
3. $a \leq b \Longrightarrow \neg_{\mathbf{2}} b \leq \neg_{\mathbf{2}} a$.
4. $a \leq \neg_{\mathbf{2}} \neg_{\mathbf{2}} a$.
5. $\neg_{\mathbf{2}} \bigvee_i a_i = \bigwedge_i \neg_{\mathbf{2}} a_i$.
6. $\bigvee_i \neg_{\mathbf{2}} a_i \leq \neg_{\mathbf{2}}(\bigwedge_i a_i)$.
7. $\Box_{\mathbf{2}} a \leq a$.
8. $\Box_{\mathbf{2}} \Box_{\mathbf{2}} a = \Box_{\mathbf{2}} a$.
9. $\neg_{\mathbf{2}} a = \Box_{\mathbf{2}} \neg_{\mathbf{2}} a$.
10. $\bigvee_i \neg_{\mathbf{2}} \Box_{\mathbf{2}} a_i = \neg_{\mathbf{2}} \bigwedge_i \Box_{\mathbf{2}} a_i$.
11. $\bigwedge_i \Box_{\mathbf{2}} a_i = \Box_{\mathbf{2}} \bigwedge_i a_i$.
12. $a \leq b \Longrightarrow \Box_{\mathbf{2}} a \leq \Box_{\mathbf{2}} b$.
13. $\Diamond_{\mathbf{2}} a = \neg_{\mathbf{2}} \Box_{\mathbf{2}} \neg_{\mathbf{2}} a$.
14. $\Box_{\mathbf{2}} a \wedge \bigvee_i b_i = \bigvee_i (\Box_{\mathbf{2}} a \wedge b_i)$; $\quad a \wedge \bigvee_i \Box_{\mathbf{2}} b_i = \bigvee_i (a \wedge \Box_{\mathbf{2}} b_i)$.

Remark 2.19. In the above theorem, all items other than 4. and 13. hold for $\to_Z$, $\neg_Z$, $\Box_Z$ associated with an arbitrary center Z of $\mathcal{L}$.

Definition 2.20. Elements of **2** is said to be **global**

$$a \in \mathcal{L} \text{ is } \textbf{global} \overset{\text{def}}{\iff} a = \Box_{\mathbf{2}} a,$$

and implication $\to_{\mathbf{2}}$ is said to be **global (or basic) implication**. Operator $\to_{\mathbf{2}}$ represents the partial order $\leq$ of the lattice:

$$a \to_{\mathbf{2}} b = \bigvee \{x \in_{\mathbf{2}} \mathbf{2} \mid a \wedge x \leq b\} = \begin{cases} 1 & a \leq b \\ 0 & a \not\leq b \end{cases}$$

2.3 Lattice valued logic L

Lattice valued logic L is a logic which characterizes the structure of complete lattice by introducing the global implication $\to_{\mathbf{2}}$ as a logical operator.

Language of L

Primitive symbols of L are the followings :

1. Free variables : $a_1, a_2, \cdots$
2. Bound variables : $x_1, x_2, \cdots$
3. Constants : $c_1, c_2, \cdots$
4. Predicate symbols : $=_{\mathbf{2}}$, $\in_{\mathbf{2}}$, which are associated with $\to_{\mathbf{2}}$.
5. Logical symbols : $\wedge$, $\vee$, $\to_{\mathbf{2}}$, $\neg_{\mathbf{2}}$, $\forall$, $\exists$
6. Parentheses : (,), [,]

Free variables and constants are called **terms**, and denoted by $t_1, t_2, \cdots$. **Atomic formula** of L are expressions of the form $t_1 =_{\mathbf{2}} t_2$ or $t_1 \in_{\mathbf{2}} t_2$ with terms t_1, t_2. **Formulas** of L are constructed from atomic formulas using logical operators. To denote formulas, we use

$$\varphi, \psi, \xi, \cdots, \ \varphi(a), \cdots.$$

Definition 2.21. **$\Box_{\mathbf{2}}$-closed formulas** are defined inductively :

1. A formula of the form $\varphi \to_{\mathbf{2}} \psi$ or $\neg_{\mathbf{2}} \varphi$ is $\Box_{\mathbf{2}}$-closed.
2. If formulas φ and ψ are $\Box_{\mathbf{2}}$-closed, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\Box_{\mathbf{2}}$-closed.
3. If a formula $\varphi(a)$ is a $\Box_{\mathbf{2}}$-closed formula with free variable a, then $\forall x \varphi(x)$ and $\exists x \varphi(x)$ are $\Box_{\mathbf{2}}$-closed.

4. $\Box_2$-closed formulas are only those obtained by (1)–(3).

$\Gamma, \Delta, \Pi, \Lambda, \cdots$ will be used to denote finite sequences of formulas ; $\overline{\varphi}, \overline{\psi}, \cdots$ to denote $\Box_2$-closed formulas ; and $\overline{\Gamma}, \overline{\Delta}, \overline{\Pi}, \overline{\Lambda}, \cdots$ to denote finite sequences of $\Box_2$-closed formulas. A formal expression of the form $\Gamma \Rightarrow \Delta$ is called a **sequent**.

Beginning sequents : Every proof starts with sequents of the form $\varphi \Rightarrow \varphi$ which are called **logical axioms**.

Structural rules :

Thinning :
$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

Contraction :
$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

Interchange :
$$\frac{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}$$

Cut :
$$\frac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \overline{\Delta}, \Lambda} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \overline{\Pi} \Rightarrow \Lambda}{\Gamma, \overline{\Pi} \Rightarrow \Delta, \Lambda}$$
$$\frac{\Gamma \Rightarrow \Delta, \overline{\varphi} \quad \overline{\varphi}, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical rules:

$\neg_2$:
$$\frac{\Gamma \Rightarrow \overline{\Delta}, \varphi}{\neg_2\varphi, \Gamma \Rightarrow \overline{\Delta}} \qquad \frac{\Gamma \Rightarrow \Delta, \overline{\varphi}}{\neg_2\overline{\varphi}, \Gamma \Rightarrow \Delta} \qquad \frac{\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}}{\overline{\Gamma} \Rightarrow \overline{\Delta}, \neg_2\varphi} \qquad \frac{\overline{\varphi}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_2\overline{\varphi}}$$

$\wedge$:
$$\frac{\varphi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \Gamma \Rightarrow \overline{\Delta}, \psi}{\Gamma \Rightarrow \overline{\Delta}, \varphi \wedge \psi}$$
$$\frac{\psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \overline{\varphi} \quad \Gamma \Rightarrow \Delta, \overline{\psi}}{\Gamma \Rightarrow \Delta, \overline{\varphi} \wedge \overline{\psi}}$$

$\vee:$

$$\frac{\varphi, \overline{\Gamma} \Rightarrow \Delta \quad \psi, \overline{\Gamma} \Rightarrow \Delta}{\varphi \vee \psi, \overline{\Gamma} \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\overline{\varphi}, \Gamma \Rightarrow \Delta \quad \overline{\psi}, \Gamma \Rightarrow \Delta}{\overline{\varphi} \vee \overline{\psi}, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$\rightarrow_2:$

$$\frac{\Gamma \Rightarrow \overline{\Delta}, \varphi \quad \psi, \overline{\Pi} \Rightarrow \Lambda}{(\varphi \rightarrow_2 \psi), \Gamma, \overline{\overline{\Pi}} \Rightarrow \overline{\Delta}, \Lambda} \qquad \frac{\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \psi}{\overline{\Gamma} \Rightarrow \overline{\Delta}, (\varphi \rightarrow_2 \psi)} \qquad \frac{\overline{\varphi}, \Gamma \Rightarrow \Delta, \overline{\psi}}{\Gamma \Rightarrow \Delta, (\overline{\varphi} \rightarrow_2 \overline{\psi})}$$

$\forall:$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta}$$

where t is any term

$$\frac{\Gamma \Rightarrow \overline{\Delta}, \varphi(a)}{\Gamma \Rightarrow \overline{\Delta}, \forall x \varphi(x)} \qquad \frac{\Gamma \Rightarrow \Delta, \overline{\varphi}(a)}{\Gamma \Rightarrow \Delta, \forall x \overline{\varphi}(x)}$$

where a is a free variable which does not occur in the lower sequent.

$\exists:$

$$\frac{\varphi(a), \overline{\Gamma} \Rightarrow \Delta}{\exists x \varphi(x), \overline{\Gamma} \Rightarrow \Delta} \qquad \frac{\overline{\varphi}(a), \Gamma \Rightarrow \Delta}{\exists x \overline{\varphi}(x), \Gamma \Rightarrow \Delta}$$

where a is a free variable which does not occur in the lower sequent.

$$\frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}$$

where t is any term

We use the following abbreviations.

$$\begin{aligned}
\top &\stackrel{\text{def}}{=} (\varphi \rightarrow_2 \varphi) \quad \text{for some } \varphi, \\
\bot &\stackrel{\text{def}}{=} \neg_2 \top \\
\Box_2 \varphi &\stackrel{\text{def}}{=} ((\varphi \rightarrow_2 \varphi) \rightarrow_2 \varphi) \\
\Diamond_2 \varphi &\stackrel{\text{def}}{=} \neg_2 \Box_2 \neg_2 \varphi \\
\varphi \leftrightarrow_2 \psi &\stackrel{\text{def}}{=} (\varphi \rightarrow_2 \psi) \wedge (\psi \rightarrow_2 \varphi) \\
u \in_2^{\Box} v &\stackrel{\text{def}}{=} \Box_2 (u \in_2 v). \\
\mathrm{L} \vdash \Gamma \Rightarrow \Delta &\stackrel{\text{def}}{\Longleftrightarrow} \Gamma \Rightarrow \Delta \text{ is provable in } \mathrm{L}
\end{aligned}$$

$\mathrm{L} \vdash \Gamma \Rightarrow \Delta$ may be written as $\vdash \Gamma \Rightarrow \Delta$, if any confusion is not likely. $\vdash \varphi \Leftrightarrow \psi$ means that "$\vdash \varphi \Rightarrow \psi$ and $\vdash \psi \Rightarrow \varphi$".

Theorem 2.22. *For formulas of φ, ψ, ξ of* L,

1. $\vdash \varphi \wedge \psi, \Gamma \Rightarrow \Delta$ *if and only if* $\vdash \varphi, \psi, \Gamma \Rightarrow \Delta$
2. $\vdash \Gamma \Rightarrow \Delta, \varphi \vee \psi$ *if and only if* $\vdash \Gamma \Rightarrow \Delta, \varphi, \psi$

3. $\vdash \Box_2\varphi \Rightarrow \varphi$; $\vdash \varphi \Rightarrow \Diamond_2\varphi$

4. $\vdash \overline{\Gamma} \Rightarrow \overline{\Delta}, \varphi$ *if and only if* $\vdash \overline{\Gamma} \Rightarrow \overline{\Delta}, \Box_2\varphi$;
   $\vdash \varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}$ *if and only if* $\vdash \Diamond_2\varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}$

5. $\vdash \overline{\varphi} \Leftrightarrow \Box_2\overline{\varphi}$

6. $\vdash \varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \psi$ *if and only if* $\vdash \overline{\Gamma} \Rightarrow \overline{\Delta}, (\varphi \rightarrow_2 \psi)$

7. $\vdash \varphi \wedge (\varphi \rightarrow_2 \psi) \Rightarrow \psi$

8. $\vdash \varphi \wedge \neg_2\varphi \Rightarrow$; $\vdash \Rightarrow \overline{\varphi} \vee \neg_2\overline{\varphi}$

9. $\vdash \varphi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \psi$ *implies* $\vdash \neg_2\psi, \overline{\Gamma} \Rightarrow \overline{\Delta}, \neg_2\varphi$

10. $\vdash \varphi \Rightarrow \neg_2\neg_2\varphi$; $\vdash \overline{\varphi} \Leftrightarrow \neg_2\neg_2\overline{\varphi}$

11. $\vdash \neg_2(\varphi \vee \psi) \Leftrightarrow (\neg_2\varphi \wedge \neg_2\psi)$; $\vdash (\neg_2\varphi \vee \neg_2\psi) \Rightarrow \neg_2(\varphi \wedge \psi)$

12. $\vdash (\overline{\varphi} \rightarrow_2 \psi) \Leftrightarrow (\overline{\varphi} \rightarrow_2 \Box_2\psi) \Leftrightarrow (\neg_2\overline{\varphi} \vee \Box_2\psi)$

13. $\vdash \overline{\varphi} \wedge (\psi_1 \vee \psi_2) \Leftrightarrow (\overline{\varphi} \wedge \psi_1) \vee (\overline{\varphi} \wedge \psi_2)$;
    $\vdash \varphi \wedge (\overline{\psi_1} \vee \overline{\psi_2}) \Leftrightarrow (\varphi \wedge \overline{\psi_1}) \wedge (\varphi \vee \overline{\psi_2})$

14. $\vdash \overline{\varphi} \wedge \exists x\psi(x) \Leftrightarrow \exists x(\overline{\varphi} \wedge \psi(x))$; $\vdash \varphi \wedge \exists x\overline{\psi}(x) \Leftrightarrow \exists x(\varphi \wedge \overline{\psi}(x))$

15. $\vdash [(\varphi \wedge \overline{\xi}) \rightarrow_2 \psi] \Rightarrow [(\neg_2\psi \wedge \overline{\xi}) \rightarrow_2 \neg_2\varphi]$

16. $\vdash \Diamond_2(\overline{\varphi} \wedge \psi) \Rightarrow \overline{\varphi} \wedge \Diamond_2\psi$

17. $\vdash \forall x\Box_2\varphi(x) \Leftrightarrow \Box_2\forall x\varphi(x)$

18. $\vdash \exists x\Diamond_2\varphi(x) \Leftrightarrow \Diamond_2\exists x\varphi(x)$

19. *If* $A(\varphi)$ *is a formula with sub-formula* φ *and* $\vdash \varphi \Leftrightarrow \psi$, *then*

$$\vdash A(\varphi) \Leftrightarrow A(\psi).$$

Soundness and completeness of lattice valued logic L was proved by M.Takano [9].

Definition 2.23. A formula φ is said to be **global** if $\vdash \varphi \Leftrightarrow \Box_2\varphi$.

2.4 Lattice valued universe $V^{\mathcal{L}}$

Let V be a universe of classical set theory ZFC. V is considered as the external universe in which lattice valued universe is constructed.

For arbitrary complete lattice $\mathcal{L}$ in V, $\mathcal{L}$-valued universe $V^{\mathcal{L}}$ is constructed by transfinite induction:

$$\begin{aligned} V^{\mathcal{L}}_\alpha &= \{u \mid \exists\beta < \alpha\, \exists\mathcal{D}u \subset V^{\mathcal{L}}_\beta (u : \mathcal{D}u \rightarrow \mathcal{L})\} \\ V^{\mathcal{L}} &= \textstyle\bigcup_{\alpha\in\mathrm{On}} V^{\mathcal{L}}_\alpha \end{aligned}$$

The least α such that $u \in V_\alpha^{\mathcal{L}}$ is called the **rank** of u.

Lattice valued set theory was formulated in Titani [12] as a set theory developed in the lattice valued universe by lattice valued logic L.

Truth values $[\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}}$ and $[\![u \in_{\mathbf{2}} v]\!]_{\mathcal{L}}$ in $V^{\mathcal{L}}$ of atomic formulas of LZFZ are given by transfinite induction on the rank of u, v.

$$\begin{aligned} [\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}} &= \bigwedge_{x \in \mathcal{D}u}(u(x) \to_{\mathbf{2}} [\![x \in_{\mathbf{2}} v]\!]_{\mathcal{L}}) \wedge \bigwedge_{x \in \mathcal{D}v}(v(x) \to_{\mathbf{2}} [\![x \in_{\mathbf{2}} u]\!]_{\mathcal{L}}) \\ [\![u \in_{\mathbf{2}} v]\!]_{\mathcal{L}} &= \bigvee_{x \in \mathcal{D}v} [\![u =_{\mathbf{2}} x]\!]_{\mathcal{L}} \wedge v(x). \end{aligned}$$

Logical operators are interpreted in $V^{\mathcal{L}}$ as the corresponding operators on $\mathcal{L}$:

$$\begin{aligned} &[\![\neg_{\mathbf{2}} \varphi]\!]_{\mathcal{L}} = \neg_{\mathbf{2}} [\![\varphi]\!]_{\mathcal{L}} \\ &[\![\varphi_1 \wedge \varphi_2]\!]_{\mathcal{L}} = [\![\varphi_1]\!]_{\mathcal{L}} \wedge [\![\varphi_2]\!]_{\mathcal{L}} \\ &[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!] \\ &[\![\varphi_1 \to_{\mathbf{2}} \varphi_2]\!]_{\mathcal{L}} = ([\![\varphi_1]\!]_{\mathcal{L}} \to_{\mathbf{2}} [\![\varphi_2]\!]_{\mathcal{L}}) \\ &[\![\forall x \varphi(x)]\!]_{\mathcal{L}} = \bigwedge_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]_{\mathcal{L}} \\ &[\![\exists x \varphi(x)]\!]_{\mathcal{L}} = \bigvee_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]_{\mathcal{L}} \end{aligned}$$

Lemma 2.24. *For $u, v \in V^{\mathcal{L}}$, $\{u_k\}_k$, $\{v_k\}_k \subset V^{\mathcal{L}}$, $b \in \mathcal{L}$ and $\{b_k\}_k \subset \mathcal{L}$,*

$$[\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}} \wedge \bigvee_k b_k = \bigvee_k ([\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}} \wedge b_k); \quad (\bigvee_k [\![u_k =_{\mathbf{2}} v_k]\!]_{\mathcal{L}}) \wedge b = \bigvee_k ([\![u_k =_{\mathbf{2}} v_k]\!]_{\mathcal{L}} \wedge b).$$

Proof. $[\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}}$ is global for every $u, v \in V^{\mathcal{L}}$. Hence, it is obvious by Theorem 2.18. □

Theorem 2.25 ([12])**.** *For $u, v, w \in V^{\mathcal{L}}$,*

1. $[\![u =_{\mathbf{2}} v \wedge v =_{\mathbf{2}} w]\!]_{\mathcal{L}} \leq [\![u =_{\mathbf{2}} w]\!]_{\mathcal{L}}$
2. $[\![u =_{\mathbf{2}} v \wedge v \in_{\mathbf{2}} w]\!]_{\mathcal{L}} \leq [\![u \in_{\mathbf{2}} w]\!]_{\mathcal{L}}$
3. $[\![u =_{\mathbf{2}} v \wedge w \in_{\mathbf{2}} v]\!]_{\mathcal{L}} \leq [\![w \in_{\mathbf{2}} u]\!]_{\mathcal{L}}$
4. $[\![u =_{\mathbf{2}} v \wedge \varphi(u)]\!]_{\mathcal{L}} \leq [\![\varphi(v)]\!]_{\mathcal{L}}$.

Definition 2.26. $u \in V^{\mathcal{L}}$ is said to be **definite** if $u(x) = [\![x \in_{\mathbf{2}} u]\!]_{\mathcal{L}}$ for all $x \in \mathcal{D}u$, and **hereditarily definite** if u is definite and each element of $\mathcal{D}u$ is hereditarily definite.

Lemma 2.27. *For any $u \in V_\alpha^{\mathcal{L}}$ there exists a hereditarily definite $v \in V_\alpha^{\mathcal{L}}$ such that $[\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}} = 1$.*

Proof. By transfinite induction on the rank of u. Let $u \in V_\alpha^{\mathcal{L}}$. Since $\mathcal{D}u \subset V_{<\alpha}^{\mathcal{L}}$, there exists a hereditarily definite $y \in V_{<\alpha}^{\mathcal{L}}$ such that $[\![x =_{\mathbf{2}} y]\!]_{\mathcal{L}} = 1$ for all $x \in \mathcal{D}u$ by induction hypothesis. Let

$$\begin{aligned} \mathcal{D}v &= \{y \in V_{<\alpha}^{\mathcal{L}} \mid y \text{ is hereditarily definite and } \exists x \in \mathcal{D}u([\![x =_{\mathbf{2}} y]\!]_{\mathcal{L}} = 1)\} \\ v(y) &= [\![y \in_{\mathbf{2}} u]\!]_{\mathcal{L}}. \end{aligned}$$

□

Definition 2.28. Bounded quantifiers $\forall x \in_2 u$ and $\exists x \in_2 u$ are defined as follows:

$$\forall x \in_2 u \varphi(x) \overset{\text{def}}{\iff} \forall x(x \in_2 u \to_2 \varphi(x)), \quad \exists x \in_2 u \varphi(x) \overset{\text{def}}{\iff} \exists x(x \in_2 u \wedge \varphi(x))$$

Lemma 2.29 (Titani [12]). *For a formula $\varphi(a)$ and $u \in V^{\mathcal{L}}$,*

$$[\![\forall x \in_2 u \varphi(x)]\!]_{\mathcal{L}} = \bigwedge_{x \in \mathcal{D}u} [\![x \in_2 u \to_2 \varphi(x)]\!]_{\mathcal{L}}, \quad [\![\exists x \in_2 u \varphi(x)]\!]_{\mathcal{L}} = \bigvee_{x \in \mathcal{D}u} [\![x \in_2 u \wedge \varphi(x)]\!]_{\mathcal{L}}.$$

Hence, if u is definite,

$$[\![\forall x \in_2 u \varphi(x)]\!]_{\mathcal{L}} = \bigwedge_{x \in \mathcal{D}u} (u(x) \to_2 [\![\varphi(x)]\!]_{\mathcal{L}}), \quad [\![\exists x \in_2 u \varphi(x)]\!]_{\mathcal{L}} = \bigvee_{x \in \mathcal{D}u} u(x) \wedge [\![\varphi(x)]\!]_{\mathcal{L}}.$$

Theorem 2.30 ([12]). *The following A1–A11 are valid in $V^{\mathcal{L}}$, that is, $[\![\mathbf{A}i]\!]_{\mathcal{L}} = 1$ for $i = 1, \cdots, 11$.*

A1. Equality $\forall u \forall v\, [u =_2 v \wedge \varphi(u) \to_2 \varphi(v)]$.

A2. Extensionality $\forall u, v\, [\forall x(x \in_2 u \leftrightarrow_2 x \in_2 v) \to_2 u =_2 v]$.

A3. Pairing $\forall u, v \exists z \forall x[x \in_2 z \leftrightarrow_2 (x =_2 u \vee x =_2 v)]$.

The set z satisfying $\forall x[x \in_2 z \leftrightarrow_2 (x =_2 u \vee x =_2 v)]$ is denoted by $\{u, v\}$.

A4. Union $\forall u \exists z \forall x[x \in_2 z \leftrightarrow_2 \exists y \in_2 u(x \in_2 y)]$.

The set z satisfying $\forall x[x \in_2 z \leftrightarrow_2 \exists y \in_2 u(x \in_2 y)]$ is denoted by $\bigcup u$.

A5. Power set $\forall u \exists z \forall x[x \in_2 z \leftrightarrow_2 x \subset_2 u]$, *where* $x \subset_2 u \overset{def}{\iff} \forall y \in_2 x(y \in_2 u)$.

The set z satisfying $\forall x[x \in_2 z \leftrightarrow_2 x \subset_2 u]$ is denoted by $\mathcal{P}(u)$.

A6. Infinity $\exists u\, [\exists x \in_2 u(\forall y \neg(y \in_2 x)) \wedge \forall x \in_2 u(x \cup \{x\} \in_2 u)]$.

A7. Separation $\forall u \exists v \forall x[x \in_2 v \leftrightarrow_2 x \in_2 u \wedge \varphi(x)]$.

The set v satisfying $\forall x[x \in_2 v \leftrightarrow_2 x \in_2 u \wedge \varphi(x)]$ is denoted by $\{x \in_2 u \mid \varphi(x)\}$.

A8. Collection $\forall u \exists v\, [\forall x \in_2 u \exists y \varphi(x, y) \to_2 \forall x \in_2 u \exists y \in_2 v \varphi(x, y)]$.

A9. $\in_2$-induction $\forall x\, (\forall y \in_2 x \varphi(y) \to_2 \varphi(x)) \to_2 \forall x \varphi(x)$.

A10. Zorn $\exists x(x \in_2 u) \wedge \mathrm{Gl}(u) \wedge \forall v\, [\mathrm{Chain}(v, u) \to_2 (\bigcup v \in_2 u)] \to_2 \exists z \mathrm{Max}(z, u)$, *where*

$$\begin{aligned} \mathrm{Gl}(u) &\overset{def}{\iff} \forall x \in_2 u\, (x \in_2^{\Box} u) \quad (i.e.\ \forall x \in_2 u\, \Box_2(x \in_2 u)), \\ \mathrm{Chain}(v, u) &\overset{def}{\iff} v \subset_2 u \wedge \forall x, y \in_2 v(x \subset_2 y \vee y \subset_2 x), \\ \mathrm{Max}\,(z, u) &\overset{def}{\iff} z \in_2 u \wedge \forall x \in_2 u(z \subset_2 x \to_2 z =_2 x). \end{aligned}$$

A11. Axiom of $\Diamond_2$ $\forall u \exists z \forall t[t \in_2 z \leftrightarrow_2 \Diamond_2(t \in_2 u)]$, *where* $\Diamond_2\varphi \overset{def}{\iff} \neg_2\Box_2\neg_2\varphi$.
The set z satisfying $\forall t[t \in_2 z \leftrightarrow_2 \Diamond_2(t \in u)]$ is denoted by $\Diamond_2 u$.

Lattice valued set theory LZFZ (Lattice valued Zermero-Fraenkel set theory with Zorn's Lemma) is obtained by adjoining the lattice valued logic L and nonlogical axioms A1–A11 of Theorem 2.30.

2.5 Quantum logic QL

Quantum logic is a logic representing the structure of complete orthomodular lattices. That is, quantum logic QL is the lattice valued logic with addtional logical operator $^\perp$ together with the following logical axioms (**C**) and (**P**):

C (Orthocomplement):

- **C1** $\varphi \Leftrightarrow \varphi^{\perp\perp}$
- **C2** $\Rightarrow \varphi \vee \varphi^{\perp}, \quad \varphi \wedge \varphi^{\perp} \Rightarrow$
- **C3** $(\varphi \to_2 \psi) \Rightarrow (\psi^{\perp} \to_2 \varphi^{\perp})$

P (Orthomodular): $(\varphi \to_2 \psi), \psi \Rightarrow \psi \wedge \varphi, \psi \wedge \varphi^{\perp}$.

If $\Gamma \Rightarrow \Delta$ is provable in QL, then we write

$$\mathrm{QL} \vdash \Gamma \Rightarrow \Delta, \quad \text{or simply} \quad \vdash \Gamma \Rightarrow \Delta.$$

Definition 2.31. Formulas φ and ψ are said to be **compatible**, if

$$\vdash \psi \Rightarrow (\psi \wedge \varphi) \vee (\psi \wedge \varphi^{\perp}).$$

The formula $\psi \to_2 (\psi \wedge \varphi) \vee (\psi \wedge \varphi^{\perp})$ is denoted by $\varphi \rfloor_\circ \psi$.

$$\varphi \rfloor_\circ \psi \overset{\text{def}}{\iff} \psi \to_2 (\psi \wedge \varphi) \vee (\psi \wedge \varphi^{\perp})$$

Lemma 2.32. *1. $\top$ and $\bot$ are global.*

2. If φ is a global formula, then $\vdash \Rightarrow \varphi \rfloor_\circ \psi$ for every formula ψ.

3. If φ is a global formula, then $\vdash \varphi^{\perp} \Leftrightarrow \neg_2\varphi$.

4. If $A(\varphi)$ is a formula with subformula φ and $\vdash \varphi \Leftrightarrow \psi$, then

$$\vdash A(\varphi) \Leftrightarrow A(\psi).$$

Proof. 1. Since $\vdash \top \Leftrightarrow (\varphi \to_2 \varphi)$ and $\vdash \bot \Leftrightarrow \neg_2\top$.

2. By Theorem 2.22.

3. $\vdash\ \Rightarrow \varphi, \varphi^{\perp}$ and $\vdash\ \varphi, \varphi^{\perp} \Rightarrow$ by axiom C2. It follows that

$$\vdash \neg_2 \varphi \Leftrightarrow \varphi^{\perp}$$

4. By induction on the complexity of the formula A.

□

Obviously we have

Theorem 2.33. *Every complete orthomodular lattice is a model of quantum logic* QL*. Especially, $P(\mathcal{H})$ is a model of* QL*.*

2.6 Quantum set theory QZFZ

Quantum set theory QZFZ (Quantum Zermero-Fraenkel set theory with Zorn's Lemma) is obtained by adjoining quantum logic QL and nonlogical axioms A1–A11 of Theorem 2.30. That is, QZFZ is a lattice valued set theory provided with logical operator $^{\perp}$ which is interpreted as $[\![\varphi^{\perp}]\!] = [\![\varphi]\!]^{\perp}$, and with logical axioms **C** and **P**.

Since $P(\mathcal{H})$ is a complete orthomodular lattice, $P(\mathcal{H})$-valued universe $V^{P(\mathcal{H})}$ is a model of quantum set theory QZFZ.

2.7 Quantum universe $V^{P(\mathcal{H})}$

2.7.1 $(\bigvee, {}^{\perp})$-preserving injections

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete orthomodular lattices, and let $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a $(\bigvee, {}^{\perp})$-preserving injection. That is,

1. $a, b \in \mathcal{L}_1,\ f(a) = f(b) \Longrightarrow a = b$,
2. $A \subset \mathcal{L}_1 \Longrightarrow\ f(\bigvee_{a\in A} a) = \bigvee_{a\in A} f(a)$,
3. $a \in \mathcal{L}_1 \Longrightarrow\ f(a^{\perp}) = (f(a))^{\perp}$.

Then $\leq$, $\bigwedge$, $\to_2$, 1, 0 are preserved. That is, $a \leq b \Longleftrightarrow f(a) \leq f(b)$ and

$$f(\bigwedge_{a\in A} a) = \bigwedge_{a\in A} f(a),\ \ f(a \to_2 b) = (f(a) \to_2 f(b)),\ \ f(1_{\mathcal{L}_1}) = 1_{\mathcal{L}_2},\ \ f(0_{\mathcal{L}_1}) = 0_{\mathcal{L}_2},$$

where $1_{\mathcal{L}_i} = \bigvee \mathcal{L}_i$ and $0_{\mathcal{L}_i} = \bigwedge \mathcal{L}_i$ for $i = 1, 2$. The $(\bigvee, {}^{\perp})$-preserving injection $f : \mathcal{L}_1 \to \mathcal{L}_2$ is extended to a mapping of universe $f : V^{\mathcal{L}_1} \to V^{\mathcal{L}_2}$, as follows :

Definition 2.34. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete orthomodular lattices, and $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a $(\bigvee, {}^{\perp})$-preserving injection, then $f : V^{\mathcal{L}_1} \to V^{\mathcal{L}_2}$ is defined inductively by

$$\begin{cases} \mathcal{D}(f(u)) = \{f(x) \mid x \in \mathcal{D}u\} \\ (f(u))(v) = \bigvee \{f[\![x \in_2 u]\!] \mid x \in \mathcal{D}u,\ [\![v =_2 f(x)]\!]_{\mathcal{L}_2} = 1\} \end{cases} \quad (2.1)$$

Theorem 2.35. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete orthomodular lattices, and $f:\mathcal{L}_1\to\mathcal{L}_2$ be a $(\bigvee,{}^\perp)$-preserving injection. Then for $u,v\in V^{\mathcal{L}_1}$,*

$$f[\![u=_2 v]\!]_{\mathcal{L}_1}=[\![f(u)=_2 f(v)]\!]_{\mathcal{L}_2}\quad \textit{and}\quad f[\![u\in_2 v]\!]_{\mathcal{L}_1}=[\![f(u)\in_2 f(v)]\!]_{\mathcal{L}_2}.$$

Proof. We proceed by transfinite induction on the rank of u,v. Let u,v be elements of $V^{\mathcal{L}_1}_\alpha$. By the induction hypothesis, $[\![f(x)=_2 f(x')]\!]_{\mathcal{L}_2}=1$ implies $[\![x=_2 x']\!]_{\mathcal{L}_1}=1$ for $x,x'\in\mathcal{D}u$. Hence we have

$$(f(u))(f(x))=f[\![x\in_2 u]\!]_{\mathcal{L}_1}\quad \text{for } x\in\mathcal{D}u.$$

$$\begin{aligned}
\because\ )\ (f(u))(f(x)) &= \bigvee\{f[\![x'\in_2 u]\!]_{\mathcal{L}_1}\mid x'\in\mathcal{D}u,\ [\![f(x)=_2 f(x')]\!]_{\mathcal{L}_2}=1\}\\
&= \bigvee\{f([\![x'\in_2 u]\!]_{\mathcal{L}_1}\wedge[\![x=_2 x']\!]_{\mathcal{L}_1})\mid x'\in\mathcal{D}u\}\\
&= f[\![x\in_2 u]\!]_{\mathcal{L}_1}
\end{aligned}$$

Therefore, for $x\in V^{\mathcal{L}}_{<\alpha}$,

$$\begin{aligned}
f[\![x\in_2 u]\!]_{\mathcal{L}_1} &= f(\bigvee_{y\in\mathcal{D}u}[\![x=_2 y]\!]_{\mathcal{L}_1}\wedge[\![y\in_2 u]\!]_{\mathcal{L}_1})\\
&= \bigvee_{y\in\mathcal{D}u} f[\![x=_2 y]\!]_{\mathcal{L}_1}\wedge f[\![y\in_2 u]\!]_{\mathcal{L}_1}\\
&= \bigvee_{y\in\mathcal{D}u}[\![f(x)=_2 f(y)]\!]_{\mathcal{L}_2}\wedge(f(u))(f(y))\\
&= [\![f(x)\in_2 f(u)]\!]_{\mathcal{L}_2}
\end{aligned}$$

It follows that

$$\begin{aligned}
f[\![u=_2 v]\!]_{\mathcal{L}_1}=1 &\iff [\![u=_2 v]\!]_{\mathcal{L}_1}=1\\
&\implies (f(u))(f(x))=f[\![x\in_2 u]\!]_{\mathcal{L}_1}\le f[\![x\in_2 v]\!]_{\mathcal{L}_1}\\
&\implies (f(u))(f(x))\le[\![f(x)\in_2 f(v)]\!]_{\mathcal{L}_2}\ \text{ for } x\in\mathcal{D}u\\
\text{Similarly}\qquad & \qquad (f(v))(f(y))\le[\![f(y)\in_2 f(u)]\!]_{\mathcal{L}_2}\ \text{ for } x\in\mathcal{D}v\\
&\implies [\![f(u)=_2 f(v)]\!]_{\mathcal{L}_2}=1.
\end{aligned}$$

$$\begin{aligned}
[\![f(u)=_2 f(v)]\!]_{\mathcal{L}_2}=1 &\implies f[\![x\in_2 u]\!]_{\mathcal{L}_1}=f(u)(f(x))\\
&\qquad\qquad \le[\![f(x)\in_2 f(v)]\!]_{\mathcal{L}_2}=f[\![x\in_2 v]\!]_{\mathcal{L}_1}\\
&\implies [\![x\in_2 u]\!]_{\mathcal{L}_1}\le[\![x\in_2 v]\!]_{\mathcal{L}_1}\ \text{ for } x\in\mathcal{D}u\\
\text{Similarly}\qquad & \qquad [\![x\in_2 v]\!]_{\mathcal{L}_1}\le[\![x\in_2 u]\!]_{\mathcal{L}_1}\ \text{ for } x\in\mathcal{D}v\\
&\implies [\![u=_2 v]\!]_{\mathcal{L}_1}=1\\
&\implies f[\![u=_2 v]\!]_{\mathcal{L}_1}=1
\end{aligned}$$

$$\therefore\quad f[\![u=_2 v]\!]_{\mathcal{L}_1}=[\![f(u)=_2 f(v)]\!]_{\mathcal{L}_2}.$$

$$\begin{aligned} f[\![u \in_2 v]\!]_{\mathcal{L}_1} &= f(\bigvee_{x\in\mathcal{D}v} [\![u =_2 x]\!]_{\mathcal{L}_1} \wedge [\![x \in_2 v]\!]_{\mathcal{L}_1}) \\ &= \bigvee_{x\in\mathcal{D}v} [\![f(u) =_2 f(x)]\!]_{\mathcal{L}_2} \wedge [\![f(x) \in_2 f(v)]\!]_{\mathcal{L}_2} = [\![f(u) \in_2 f(v)]\!]_{\mathcal{L}_2}. \end{aligned}$$

□

Theorem 2.36. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete orthomodular lattices, and $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a $(\bigvee, ^\perp)$-preserving injection. If $\varphi(x_1, x_2, \cdots, x_n)$ is a bounded formula of* QZFZ*, i.e. constructed by logical operators $\vee$, $\wedge$, $\neg_2$, $^\perp$, $\to_2$, bounded quantifiers $\forall x \in_2 u$, $\exists x \in_2 u$, and constants $u, u_1, u_2, \cdots, u_n \in V^{\mathcal{L}_1}$, then*

$$f[\![\varphi(u_1, u_2, \cdots, u_n)]\!]_{\mathcal{L}_1} = [\![\varphi(f(u_1), f(u_2), \cdots, f(u_n))]\!]_{\mathcal{L}_2}. \qquad (2.2)$$

Especially, if f is a bijection preserving $\bigvee$ and $^\perp$, then the equation (2.2) holds for every formula φ of QZFZ.

Proof. We prove only the case that φ is $\forall x \in_2 u \psi(x)$, by induction on the complexity of φ.

$$\begin{aligned} & f[\![\forall x \in_2 u \psi(x, u_1, \cdots u_n)]\!]_{\mathcal{L}_1} = f(\bigwedge_{x\in\mathcal{D}u} [\![x \in_2 u \to_2 \psi(x, u_1, \cdots u_n)]\!]_{\mathcal{L}_1}) \\ &= \bigwedge_{x\in\mathcal{D}u} (f[\![x \in_2 u]\!]_{\mathcal{L}_1} \to_2 f[\![\psi(x, u_1, \cdots u_n)]\!]_{\mathcal{L}_1}) \\ &= \bigwedge_{f(x)\in\mathcal{D}f(u)} ([\![f(x) \in_2 f(u)]\!]_{\mathcal{L}_2} \to_2 [\![\psi(f(x), f(u_1), \cdots f(u_n))]\!]_{\mathcal{L}_2} \\ &= [\![\forall x \in_2 f(u)\, \psi(x, f(u_1), \cdots f(u_n))]\!]_{\mathcal{L}_2}. \end{aligned}$$

□

Corollary 2.37. *For each $U \in \mathcal{U}$, if φ is a bounded formula of* QZFZ*, then*

$$[\![\varphi(u_1, u_2, \cdots, u_n)]\!]_{B_U} = [\![\varphi(u_1, u_2, \cdots, u_n)]\!]_{P(\mathcal{H})}. \qquad (2.3)$$

2.7.2 Sheaf $\mathrm{Sh}_{\mathcal{U}}(V^B)$ of Boolean valued universe over $\mathcal{U}$

By Theorem 2.13, $P(\mathcal{H})$ is the union of mutually isomorphic Boolean $(\bigvee, ^\perp)$-sublattices.

$$P(\mathcal{H}) = \bigcup_{U\in\mathcal{U}} B_U, \quad \text{where } B_U = \{\sigma_U(p) \mid p \in B\}.$$

The identity mapping $B_U \to P(\mathcal{H})$ is a $(\bigvee, ^\perp)$-preserving injection. Hence, each V^{B_U} is embedded in $V^{P(\mathcal{H})}$, and $\{V^{B_U} \mid U \in \mathcal{U}\}$ forms a sheaf of Boolean valued universe, which is denoted by $\mathrm{Sh}_{\mathcal{U}}(V^B)$.

$$\mathrm{Sh}_{\mathcal{U}}(V^B) = <\widetilde{V^B}, \pi>, \quad \text{where}$$

$$\widetilde{V^B} \stackrel{\text{def}}{=} \{< U, \sigma_U(t) > \mid U \in \mathcal{U},\ t \in V^B\},$$

$$\Gamma(X,t) \stackrel{\text{def}}{=} \{< U, \sigma_U(t) > \mid U \in X\}, \quad \text{for } X \in \mathcal{O}(\mathcal{U}) \text{ and } t \in V^B$$

$\widetilde{V^B}$ is a topological space with open basis

$$\{\Gamma(X,t) \mid X \in \mathcal{O}(\mathcal{U}),\ t \in V^B\}.$$

$\pi :< U, \sigma_U(t) > \mapsto U$ is a homeomorphism.

2.7.3 Takeuti's quantum set theory

In [10] and [11], G.Takeuti developed a quantum set theory in $V^{P(\mathcal{H})}$ with logical operators $\wedge$, $\vee$, $^\perp$, $\forall$, $\exists$, and atomic formulas of the form $u \in v$ or $u = v$, where he used an implication defined in terms of $\wedge$, $\vee$ and $^\perp$. We denote his implication by $\to_{\mathbf{T}}$:

$$(\varphi \to_{\mathbf{T}} \psi) \stackrel{\text{def}}{\iff} (\varphi^\perp \vee (\varphi \wedge \psi)).$$

If $p, q, r \in P(\mathcal{H})$ are mutually compatible, then

$$p \leq (q \to_{\mathbf{T}} r) \iff p \wedge q \leq p \wedge r \iff (p \wedge q \to_{\mathbf{2}} p \wedge r) = 1.$$

So, $\to_{\mathbf{T}}$ can be considered as a local implication.

Let $\mathcal{L}$ be a complete orthomodular lattice. Atomic formulas $u \in v$, $u = v$ of Takeuti's quantum set theory are interpreted in $V^{\mathcal{L}}$ as follows, where $[\![\varphi]\!]^T$ denotes the Takeuti's interpretation of formula φ.

$$\begin{aligned}
[\![u = v]\!]^T &= \bigwedge_{x \in \mathcal{D}u} (u(x) \to_{\mathbf{T}} [\![x \in v]\!]^T) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to_{\mathbf{T}} [\![x \in u]\!]^T) \\
[\![u \in v]\!]^T &= \bigvee_{x \in \mathcal{D}v} [\![u = x]\!]^T \wedge v(x).
\end{aligned}$$

Logical operators $\wedge$, $\vee$, $^\perp$, $\forall$ and $\exists$ are interpreted as usual. Bounded quantifiers are interpreted as follows:

$$\begin{aligned}
[\![\forall x \in u \varphi(x)]\!]^T &= \bigwedge_{x \in \mathcal{D}u} (u(x) \to_{\mathbf{T}} [\![\varphi(x)]\!]^T), \\
[\![\exists x \in u \varphi(x)]\!]^T &= \bigvee_{x \in \mathcal{D}u} (u(x) \wedge [\![\varphi(x)]\!]^T).
\end{aligned}$$

Definition 2.38. Formulas $u =_{\mathbf{T}} v$, $u \in_{\mathbf{T}} v$, $\forall x \in_{\mathbf{T}} u \varphi(x)$, and $\exists x \in_{\mathbf{T}} u \varphi(x)$ of QZFZ corresponding to $\to_{\mathbf{T}}$ are defined recursively [1] by :

[1] Recursive definition with respect to $\in_{\mathbf{2}}$ is guaranteed in LZFZ (cf.Titani[12]). So is in QZFZ.

$$
\begin{aligned}
u =_{\mathbf{T}} v &\overset{\text{def}}{\Longleftrightarrow} \forall x(x \in_{\mathbf{2}} u \to_{\mathbf{T}} x \in_{\mathbf{T}} v) \wedge \forall x(x \in_{\mathbf{2}} v \to_{\mathbf{T}} x \in_{\mathbf{T}} u),\\
u \in_{\mathbf{T}} v &\overset{\text{def}}{\Longleftrightarrow} \exists x(u =_{\mathbf{T}} x \ \wedge \ x \in_{\mathbf{2}} v),\\
\forall x \in_{\mathbf{T}} u \varphi(x) &\overset{\text{def}}{\Longleftrightarrow} \forall x(x \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(x)),\\
\exists x \in_{\mathbf{T}} u \varphi(x) &\overset{\text{def}}{\Longleftrightarrow} \exists x(x \in_{\mathbf{2}} u \wedge \varphi(x)).
\end{aligned}
$$

Lemma 2.39. *Let $\mathcal{L}$ be a complete orthomodular lattice. If $u \in V^{\mathcal{L}}$ and φ is a formula of* QZFZ, *then*

$$
[\![\forall x \in_{\mathbf{T}} u \varphi(x)]\!]_{\mathcal{L}} = \bigwedge_{x \in \mathcal{D}u} [\![x \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(x)]\!]_{\mathcal{L}},
$$

$$
[\![\exists x \in_{\mathbf{T}} u \varphi(x)]\!]_{\mathcal{L}} = \bigvee_{x \in \mathcal{D}u} [\![x \in_{\mathbf{2}} u \wedge \varphi(x)]\!]_{\mathcal{L}} = [\![\exists x \in_{\mathbf{2}} u \varphi(x)]\!]_{\mathcal{L}}.
$$

Proof. If $[\![x \in_{\mathbf{2}} u]\!] = 0$ then $[\![x \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(x)]\!] = 1$ and $[\![x \in_{\mathbf{2}} u \wedge \varphi(x)]\!] = 0$.
If $[\![x \in_{\mathbf{2}} u]\!] \neq 0$, then there exists $t \in \mathcal{D}u$ such that $[\![x =_{\mathbf{2}} t]\!] = 1$. It follows that

$$
\begin{aligned}
[\![\forall x \in_{\mathbf{T}} u \varphi(x)]\!] &= \bigwedge_{x \in V^{\mathcal{L}}} [\![x \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(x)]\!]\\
&= \bigwedge_{[\![x \in_{\mathbf{2}} u]\!] \neq 0} [\![x \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(x)]\!] = \bigwedge_{t \in \mathcal{D}u} [\![t \in_{\mathbf{2}} u \to_{\mathbf{T}} \varphi(t)]\!].
\end{aligned}
$$

$$
\begin{aligned}
[\![\exists x \in_{\mathbf{T}} u \varphi(x)]\!] &= \bigvee_{x \in V^{\mathcal{L}}} [\![x \in_{\mathbf{2}} u \wedge \varphi(x)]\!]\\
&= \bigvee_{[\![x \in_{\mathbf{2}} u]\!] \neq 0} [\![x \in_{\mathbf{2}} u \wedge \varphi(x)]\!] = \bigvee_{t \in \mathcal{D}u} [\![t \in_{\mathbf{2}} u \wedge \varphi(t)]\!].
\end{aligned}
$$

□

Lemma 2.40. *If $\mathcal{L}$ is a complete orthomodular lattice and $u, v \in V^{\mathcal{L}}$ are hereditarily definite, then*

$$
[\![u = v]\!]^T = [\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}} \quad \textit{and} \quad [\![u \in v]\!]^T = [\![u \in_{\mathbf{T}} v]\!]_{\mathcal{L}}.
$$

Proof. By transfinite induction on the rank of u, v. Suppose $u, v \in V^{\mathcal{L}}_{\alpha}$. If $t \in \mathcal{D}u \subset V^{\mathcal{L}}_{<\alpha}$ then, by using the induction hypothesis,

$$
[\![t \in_{\mathbf{T}} v]\!]_{\mathcal{L}} = \bigvee_{s \in \mathcal{D}v} [\![t =_{\mathbf{T}} s]\!]_{\mathcal{L}} \wedge [\![s \in_{\mathbf{2}} v]\!]_{\mathcal{L}} = \bigvee_{s \in \mathcal{D}v} [\![t = s]\!]^T \wedge v(s) = [\![t \in v]\!]^T.
$$

Hence, using Lemma 2.39,

$$\begin{aligned}
[\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}} &= [\![\forall x(x \in_{\mathbf{2}} u \to_{\mathbf{T}} x \in_{\mathbf{T}} v)]\!]_{\mathcal{L}} \wedge [\![\forall x(x \in_{\mathbf{2}} v \to_{\mathbf{T}} x \in_{\mathbf{T}} u)]\!]_{\mathcal{L}} \\
&= \bigwedge_{x \in \mathcal{D}u} [\![x \in_{\mathbf{2}} u \to_{\mathbf{T}} x \in_{\mathbf{T}} v]\!]_{\mathcal{L}} \wedge \bigwedge_{x \in \mathcal{D}v} [\![x \in_{\mathbf{2}} v \to_{\mathbf{T}} x \in_{\mathbf{T}} u]\!] \\
&= \bigwedge_{x \in \mathcal{D}u} (u(x) \to_{\mathbf{T}} [\![x \in v]\!]^T) \wedge \bigwedge_{x \in \mathcal{D}v} (v(x) \to_{\mathbf{T}} [\![x \in u]\!]^T) \\
&= [\![u = v]\!]^T.
\end{aligned}$$

$$\begin{aligned}
[\![u \in_{\mathbf{T}} v]\!] &= [\![\exists x(u =_{\mathbf{T}} x \wedge x \in_{\mathbf{2}} v)]\!]_{\mathcal{L}} = \bigvee_{x \in \mathcal{D}v} [\![u =_{\mathbf{T}} x]\!]_{\mathcal{L}} \wedge [\![x \in_{\mathbf{2}} v]\!]_{\mathcal{L}} \\
&= \bigvee_{x \in \mathcal{D}v} [\![u = x]\!]^T \wedge v(x) = [\![u \in v]\!]^T.
\end{aligned}$$

□

It follows that the Takeuti's quantum set theory is interpreted in QZFZ:

Theorem 2.41. *If φ is a formula of Takeuti's quantum set theory in which all constants are hereditarily definite and $\varphi^{(T)}$ be the formula of* QZFZ *obtained from φ by replacing $=$, $\in$, $\to$, $\forall x \in u$, $\exists x \in u$ by $=_{\mathbf{T}}$, $\in_{\mathbf{T}}$, $\to_{\mathbf{T}}$, $\forall x \in_{\mathbf{T}} u$, $\exists x \in_{\mathbf{T}} u$ respectively, then*

$$[\![\varphi]\!]^T = [\![\varphi^{(T)}]\!]_{\mathcal{L}}.$$

Proof. By Lemma 2.40 and induction on the complexity of φ. If φ is $\forall x \in u \psi(x)$ or $\exists x \in u \psi(x)$, then by induction hypothesis,

$$\begin{aligned}
[\![\forall x \in u\, \psi(x)]\!]^T &= \bigwedge_{x \in \mathcal{D}u} (u(x) \to_{\mathbf{T}} [\![\psi(x)]\!]^T) \\
&= \bigwedge_{x \in \mathcal{D}u} (u(x) \to_{\mathbf{T}} [\![\psi(x)^{(T)}]\!]_{\mathcal{L}}) = [\![\varphi^{(T)}]\!]_{\mathcal{L}}, \\
[\![\exists x \in u\, \psi(x)]\!]^T &= \bigvee_{x \in \mathcal{D}u} (u(x) \wedge [\![\psi(x)]\!]^T) = \bigvee_{x \in \mathcal{D}u} (u(x) \wedge [\![\psi(x)^{(T)}]\!]_{\mathcal{L}}) = [\![\varphi^{(T)}]\!]_{\mathcal{L}}.
\end{aligned}$$

Other cases are obvious. □

3 Quantale valued set theory

3.1 Quantale

Definition 3.1. A complete lattice Q with an associative binary operation & is called a **quantale** whenever the operation & is distributive over arbitrary joins:

$$a \,\&\, \bigvee S = \bigvee \{a \& s \mid s \in S\}, \quad (\bigvee S) \,\&\, a = \bigvee \{s \& a \mid s \in S\} \quad \text{for } a \in Q,\ S \subset Q$$

Then, for elements a, b, c of a quantale Q,

$$\text{if } a \leqslant b, \text{ then } \ a\&c \leqslant b\&c \ \text{ and } \ c\&a \leqslant c\&b.$$

Definition 3.2. A quantale Q is said to be **unital** if there exists a unit element $\mathfrak{e}$ in Q such that

$$\mathfrak{e}\&a = a = a\&\mathfrak{e} \quad \text{for all } a \in Q.$$

An element $a \in Q$ of a quantale Q is said to be **right-sided** if $a\&1 \leq a$; **left-sided** if $1\&a \leq a$. The set of right-sided elements of Q is denoted by $R(Q)$; the set of left-sided elements is denoted by $L(Q)$.

$$R(Q) \stackrel{\text{def}}{=} \{a \in Q \mid a\&1 \leqslant a\} \qquad L(Q) \stackrel{\text{def}}{=} \{a \in Q \mid 1\&a \leqslant a\}$$

Definition 3.3. A subset S of a quantale Q is said to be a **sub-quantale** of Q if S is a quantale with respect to $\bigvee$ and $\&$ on Q, i.e.

$$\forall A \subset S (\bigvee A \in S) \quad \text{and} \quad \forall a, b \in S\, (a\&b \in S).$$

Theorem 3.4 ([3]). *$R(Q)$ and $L(Q)$ are sub-quantales of a quantale Q such that $\bigvee R(Q) = 1$ and $\bigvee L(Q) = 1$.*

Definition 3.5. Unary operator * on a quantale Q is called an **involution** if

1. $a^{**} = a$,
2. $(a\&b)^* = b^*\&a^*$,
3. $(\bigvee_\alpha a_\alpha)^* = \bigvee_\alpha a_\alpha^*$,

for each $a, b \in Q$ and $\{a_\alpha\}_\alpha \subset Q$. A quantale provided with an involution * is called an **involutive quantale**.

Definition 3.6. For elements a, b of an involutive quantale Q, if $a^*\&b = 0$, then a, b are said to be **orthogonal**, in symbols $a \perp b$. $a^\perp$ is defined by $\bigvee\{x \in Q \mid a \perp x\}$.

$$a \perp b \stackrel{\text{def}}{\Longleftrightarrow} a^*\&b = 0, \qquad a^\perp \stackrel{\text{def}}{=} \bigvee\{x \in Q \mid a^*\&x = 0\}.$$

Obviously, $a^\perp \in R(Q)$ for $a \in Q$.

Definition 3.7. A unital and involutive quantale $Q = (Q, \leq, \bigvee, \&, ^*, \mathfrak{e})$ is called a **Gelfand quantale** if

$$a\&a^*\&a = a \quad \text{for all } a \in R(Q).$$

Lemma 3.8 (Mulvey [3]). *Gelfand quantale Q has the following properties: For $a, a_i, b \in Q$,*

1. $a \leq b \Longleftrightarrow a^* \leq b^*$

2. $(\bigwedge_i a_i)^* = \bigwedge_i a_i^*$

3. $1^* = 1$, $0^* = 0$, $\mathfrak{e}^* = \mathfrak{e}$

4. $a^* \& a^\perp = 0$

5. $a \leqslant a^{\perp\perp}$

6. $a \leq b \Rightarrow b^\perp \leq a^\perp$

7. $(\bigvee_i a_i)^\perp = \bigwedge_i a_i^\perp$

8. *If* $a \in R(Q)$*, then* $a \& a = a$; $a \& 1 = a$; $a \leq a^* \& a$

9. *If* $a, b \in R(Q)$*, then* $a \wedge b \leqslant a^* \& b$

10. *If* $a \in R(Q)$*, then* $a \wedge a^\perp = 0$

3.2 Hilbert quantale

Theorem 3.9 (Mulvey [3]). *For a complete orthocomplemented lattice $\mathcal{L}$, let $Q(\mathcal{L})$ be the set of sup-preserving mappings from $\mathcal{L}$ to itself, with the supremum given by the pointwise ordering of mappings, with the multiplication corresponding to composition of mappings, and with the unit given by the identity mapping:*

$$\begin{aligned}
Q(\mathcal{L}) &\overset{def}{=} \left\{ \varphi : \mathcal{L} \to \mathcal{L} \mid \forall A \subset \mathcal{L} \left(\varphi(\bigvee A) = \bigvee_{x \in A} \varphi(x) \right), \right\} \\
\varphi \leqslant \psi &\overset{def}{\Longleftrightarrow} \varphi(x) \leqslant \psi(x) \quad \text{for all } x \in \mathcal{L} \\
(\bigvee_\alpha \varphi_\alpha)(x) &\overset{def}{=} \bigvee_\alpha \varphi_\alpha(x) \quad \text{for all } x \in \mathcal{L} \\
(\varphi \& \psi)(x) &\overset{def}{=} \psi(\varphi(x)) \quad \text{for all } x \in \mathcal{L} \\
\mathfrak{e}(x) &\overset{def}{=} x \quad \text{for all } x \in \mathcal{L} \\
\varphi^*(x) &\overset{def}{=} (\bigvee \{ t \in \mathcal{L} \mid \varphi(t) \leqslant x^\perp \})^\perp \quad \text{for all } x \in \mathcal{L}. \\
1(x) &= \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise,} \end{cases} \\
0(x) &= 0 \quad \text{for all } x \in \mathcal{L}.
\end{aligned}$$

Then $Q(\mathcal{L})$ is a unital and involutive quantale.

Definition 3.10 (Mulvey [3]). The quantale $Q(\mathcal{L})$ defined in Theorem 3.9 is called a **quantale of endomorphisms of $\mathcal{L}$**.

Definition 3.11 (Mulvey [3]). Endomorphisms λ_x and κ_x of complete orthocomplemented lattice $\mathcal{L}$ are defined for each $x \in \mathcal{L}$ by

$$\lambda_x(y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } y \le x, \\ 1, & \text{otherwise,} \end{cases} \qquad \kappa_x(y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } y = 0, \\ x, & \text{otherwise.} \end{cases}$$

Theorem 3.12 (Mulvey [3]). *Let $Q(\mathcal{L})$ be the quantale of endomorphisms of a complete orthocomplemented lattice $\mathcal{L}$. Then*

1. *If $x \in \mathcal{L}$ then $\lambda_x,\, \kappa_x \in \mathcal{Q}(\mathcal{L})$.*
2. *$R(Q(\mathcal{L})) = \{\lambda_x \mid x \in \mathcal{L}\}$*
3. *$\lambda_x^* = \kappa_{x^\perp}$ for $x \in \mathcal{L}$.*
4. *$\kappa_x^* = \lambda_{x^\perp}$ for $x \in \mathcal{L}$.*
5. *$\lambda_x^\perp = \lambda_{x^\perp}$ for $x \in \mathcal{L}$.*
6. *$\lambda_{(\bigvee_\alpha x_\alpha)^\perp} = \bigvee_\alpha \lambda_{x_\alpha^\perp}$ for $\{x_\alpha\}_\alpha \subset \mathcal{L}$.*
7. *$\kappa_{x^\perp} = (\kappa_x)^{*\perp*}$ for $x \in \mathcal{L}$.*
8. *$L(Q(\mathcal{L})) = \{\kappa_x \mid x \in \mathcal{L}\}$*
9. *$\lambda_x \& \lambda_x^* \& \lambda_x = \lambda_x$ for $x \in \mathcal{L}$.*
10. *$< Q(\mathcal{L}), \le, \bigvee, \&, *, \mathfrak{e} >$ is a Gelfand quantale.*
11. *$x = y \iff \lambda_x = \lambda_y \iff \kappa_x = \kappa_y$*

Definition 3.13. $\chi_R : \mathcal{L} \to R(Q(\mathcal{L}))$ and $\chi_L : \mathcal{L} \to L(Q(\mathcal{L}))$ are defined by

$$\chi_R(x) \stackrel{\text{def}}{=} \lambda_{x^\perp}, \quad \chi_L(x) \stackrel{\text{def}}{=} \kappa_x \quad \text{for } x \in \mathcal{L}.$$

Corollary 3.14. *Let $Q(\mathcal{L})$ be the quantale of endomorphisms of a complete orthocomplemented lattice $\mathcal{L}$. Then, $\chi_R :< \mathcal{L}, \bigvee, ^\perp > \cong < R(Q(\mathcal{L})), \bigvee, ^\perp >$, and $\chi_L :< \mathcal{L}, \bigvee, ^\perp > \cong < L(Q(\mathcal{L})), \bigvee, ^{*\perp*} >$, i.e.*

1. *$\chi_R(\bigvee_\alpha x_\alpha) = \bigvee_\alpha \chi_R(x_\alpha)$*
2. *$\chi_R(x^\perp) = (\chi_R(x))^\perp$*
3. *$\chi_L(\bigvee_\alpha x_\alpha) = \bigvee_\alpha \chi_L(x_\alpha)$*
4. *$\chi_L(x^\perp) = (\chi_L(x))^{*\perp*}$*

In this paper, we use term ‘Hilbert quantale’ to mean a Gelfand quantale which is isomorphic to a quantale of endomorphisms of a complete orthomodular lattice $\mathcal{L}$, whereas Mulvey used the term in [3] to mean Gelfand quantale which is isomorphic to a quantale of endomorphisms of complete orthocomplemented lattice.

Definition 3.15. A Gelfand quantale Q (cf. Definition 3.7) is said to be a **Hilbert quantale** if the following conditions are satisfied:

1. If $a \in R(Q)$, then $a = a^{\perp\perp}$,

2. If $a, b \in R(Q)$ and $a \leq b$, then $b = (b \wedge a) \vee (b \wedge a^{\perp})$.

3.3 Quantum system $(P(\mathcal{H}), \mathcal{Q})$

$\mathcal{Q} =< Q(P(\mathcal{H})), \leq, \bigvee, \&, *, \mathfrak{e} >$ is a Hilbert quantale (Definition 3.15). where

$$\begin{aligned}
Q(P(\mathcal{H})) &\overset{\text{def}}{=} \{\varphi : P(\mathcal{H}) \to P(\mathcal{H}) \mid \forall A \subset \mathcal{L}\left(\varphi(\bigvee A) = \bigvee_{x \in A} \varphi(x)\right)\} \\
\varphi \leqslant \psi &\overset{\text{def}}{\Longleftrightarrow} \varphi(x) \leqslant \psi(x) \quad \text{for all } x \in P(\mathcal{H}) \\
\left(\bigvee_\alpha \varphi_\alpha\right)(x) &\overset{\text{def}}{=} \bigvee_\alpha \varphi_\alpha(x) \quad \text{for all } x \in P(\mathcal{H}) \\
(\varphi \& \psi)(x) &\overset{\text{def}}{=} \psi(\varphi(x)) \quad \text{for all } x \in P(\mathcal{H}) \\
\mathfrak{e}(x) &\overset{\text{def}}{=} x \quad \text{for all } x \in P(\mathcal{H}) \\
\varphi^*(x) &\overset{\text{def}}{=} \left(\bigvee \{t \in P(\mathcal{H}) \mid \varphi(t) \leqslant x^{\perp}\}\right)^{\perp} \quad \text{for all } x \in P(\mathcal{H}) \\
R(\mathcal{Q}) &\overset{\text{def}}{=} \{a \in \mathcal{Q} \mid a \& 1 \leq a\} \\
L(\mathcal{Q}) &\overset{\text{def}}{=} \{a \in \mathcal{Q} \mid 1 \& a \leq a\}.
\end{aligned}$$

We define operators $\to_{\mathbf{2}}$, $^{\perp}$, $\to_{\mathbf{T}}$ as follows.

$$\text{For } a, b \in \mathcal{Q}, \qquad (a \to_{\mathbf{2}} b) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } a(x) \leqslant b(x) \text{ for all } x \in P(\mathcal{H}) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{For } a \in \mathcal{Q}, \qquad a^{\perp} \overset{\text{def}}{=} \bigvee \{b \in \mathcal{Q} \mid a^* \& b = 0\}. \text{ Then } a^{\perp} \in R(\mathcal{Q}).$$

$$\text{For } a, b \in R(\mathcal{Q}), \qquad (a \to_{\mathbf{T}} b) \overset{\text{def}}{=} a^{\perp} \vee (a \wedge b).$$

$$\text{for } a, b \in L(\mathcal{Q}), \qquad (a \to_{\mathbf{T}} b) \overset{\text{def}}{=} {}^{\perp}a \vee (a \wedge b), \quad \text{where } {}^{\perp}a \overset{\text{def}}{=} a^{*\perp*}.$$

3.3.1 Embeddings χ_R, χ_L of $P(\mathcal{H})$ into $\mathcal{Q}$

$P(\mathcal{H})$ is embedded in $\mathcal{Q}$ by $\chi_R : x \mapsto \lambda_{x^{\perp}} \in R(\mathcal{Q})$, and $\chi_L : x \mapsto \kappa_x \in L(\mathcal{Q})$, where

$$\lambda_x(y) = \begin{cases} 0, & \text{if } y \leq x, \\ 1, & \text{otherwise,} \end{cases} \qquad \kappa_x(y) = \begin{cases} 0, & \text{if } y = 0, \\ x, & \text{otherwise.} \end{cases}$$

$$\chi_R : P(\mathcal{H}) \cong R(\mathcal{Q}) \subset \mathcal{Q}, \qquad \chi_L : P(\mathcal{H}) \cong L(\mathcal{Q}) \subset \mathcal{Q} \quad \text{(Corollary 3.14).}$$

Theorem 3.16. *If $x \in P(\mathcal{H})$ and $a \in \mathcal{Q} = Q(P(\mathcal{H}))$, then*

$$\kappa_{a(x)} = \kappa_x \& a, \quad \lambda_{(a(x))^\perp} = a^* \& \lambda_{x^\perp}.$$

Proof. $(\kappa_x \& a)(y) = a(\kappa_x(y)) = \begin{cases} 0, & \text{if } y = 0 \\ a(x), & \text{otherwise} \end{cases} = \kappa_{a(x)}(y)$ □

A quantum proposition is represented by element p of $P(\mathcal{H})$, hence represented in $\mathcal{Q}$ by $\lambda_{p^\perp} \in R(\mathcal{Q})$ and $\kappa_p \in L(\mathcal{Q})$. The pair $(P(\mathcal{H}), \mathcal{Q})$ is called a **quantum system**.

Lemma 3.17. *Let $U \in \mathcal{U}$ and $\sigma_U(p) = UpU^*$ for $p \in P(\mathcal{H})$. σ_U is an automorphism of complete orthomodular lattice $P(\mathcal{H})$, hence $\sigma_U \in \mathcal{Q}$, and*

1. $(\sigma_U)^* = \sigma_{U^*}$.
2. $\chi_R(\sigma_U(p)) = \lambda_{(\sigma_U(p))^\perp} = (\sigma_U)^* \& \lambda_{p^\perp}$.
3. $\chi_L(\sigma_U(p)) = \kappa_{(\sigma_U(p))} = \kappa_p \& \sigma_U$.

Proof. 1. For $t \in P(\mathcal{H})$,

$$\begin{aligned} (\sigma_U)^*(p) \le t^\perp \quad &\iff \quad (\sigma_U)(t) \le p^\perp \iff UtU^* \le p^\perp \\ &\iff \quad t \le U^* p^\perp U = (U^* p U)^\perp \\ &\iff \quad (\sigma_{U^*})(p) \le t^\perp \end{aligned}$$

2. and 3. follows from Theorem 3.16.

□

3.3.2 Mappings χ^U_{RL}, χ^U_{LR} of B_U into $\mathcal{Q}$

$P(\mathcal{H}) = \bigcup_{U \in \mathcal{U}} B_U$, where $B_U = \{\bigvee_{j \in K} \sigma_U(p_j) \mid K \subset J\}$, by Theorem 2.13. Since $q = \bigvee_{j \in K} \sigma_U(p_j) \iff K = \{j \in J \mid \sigma_U(p_j) \le q\}$ for $U \in U$, the expression of q in B_U is unique.

Theorem 3.18. *If $U \in \mathcal{U}$ and $p, q \in B_U \subset P(\mathcal{H})$, then*

$$\chi_L(p) \& \chi_R(q) = \begin{cases} 1 & \text{if } p \wedge q \neq 0, \\ 0 & \text{if } p \wedge q = 0. \end{cases}$$

Proof. Let $p = \bigvee_{j \in K} q_j$, $q = \bigvee_{j \in K'} q_j$, where $q_j = \sigma_U(p_j)$ for each $j \in J$. If $r \neq 0$, then

$$(\chi_L(q_j) \,\&\, \chi_R(q_k))(r) = \lambda_{q_k^\perp}(\kappa_{q_j}(r)) = \lambda_{q_k^\perp}(q_j) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases}$$

Hence,

$$\begin{aligned}\chi_L(p)\&\chi_R(q) &= \bigvee_{j\in K}\chi_L(q_j)\&\bigvee_{k\in K'}\chi_R(q_k) = \bigvee_{j\in K\cap K'}(\chi_L(q_j)\&\chi_R(q_j)) \\ &= \begin{cases} 0 & \text{if } K\cap K'=\emptyset, \\ 1 & \text{if } K\cap K'\neq\emptyset. \end{cases}\end{aligned}$$

□

Definition 3.19. Let $U\in\mathcal{U}$ and $q_j=\sigma_U(p_j)$ for each $j\in J$.
For $q=\bigvee_{j\in K} q_j\in B_U$, we define $\chi^U_{RL}(q)$ and $\chi^U_{LR}(q)$ as follows.

$$\begin{aligned}\chi^U_{RL}(q) &\stackrel{\text{def}}{=} \bigvee\nolimits_{j\in K}(\chi_R(q_j)\&\chi_L(q_j)) \\ \chi^U_{LR}(q) &\stackrel{\text{def}}{=} \bigvee\nolimits_{j\in K}(\chi_L(q_j)\&\chi_R(q_j))\end{aligned}$$

Lemma 3.20. *Let $U\in\mathcal{U}$ and $q_j=\sigma_U(p_j)$ for each $j\in J$. If $j,k\in J$,*

1. $\chi^U_{RL}(q_j)\&\chi^U_{RL}(q_k)=\begin{cases}\chi^U_{RL}(q_j) & \text{if } j=k,\\ 0 & \text{if } j\neq k.\end{cases}$

2. $\chi^U_{RL}(q_j)\&\chi_R(q_k)=\begin{cases}\chi_R(q_j) & j=k,\\ 0 & j\neq k.\end{cases}$

3. $\chi_L(q_j)\&\chi^U_{RL}(q_k)=\begin{cases}\chi_L(q_j) & j=k,\\ 0 & j\neq k.\end{cases}$

Proof. 1. Since $\chi_R(q_j)\in R(\mathcal{Q})$,

$$\begin{aligned}\chi^U_{RL}(q_j)\&\chi^U_{RL}(q_k) &= \chi_R(q_j)\&\chi_L(q_j)\,\&\,\chi_R(q_k)\&\chi_L(q_k) \\ &= \begin{cases}0 & j\neq k,\\ \chi_{RL}(q_j) & j=k.\end{cases}\end{aligned}$$

2. $\chi^U_{RL}(q_j)\&\chi_R(q_k)=\chi_R(q_j)\&\left(\chi_L(q_j)\&\chi_R(q_k)\right)=\begin{cases}\chi_R(q_j) & j=k,\\ 0 & j\neq k.\end{cases}$

3. $\chi_L(q_j)\&\chi^U_{RL}(q_k)=\left(\chi_L(q_j)\&\chi_R(q_k)\right)\&\chi_L(q_k)=\begin{cases}\chi_L(q_j) & j=k,\\ 0 & j\neq k.\end{cases}$

□

Theorem 3.21. *Let $U\in\mathcal{U}$ and $p,\ q,\ q_\alpha\in B_U$ for $\alpha\in A$.*

1. $p\le q\iff \chi^U_{RL}(p)\le\chi^U_{RL}(q)$.
2. $\chi^U_{RL}(\bigvee_{\alpha\in A}q_\alpha)=\bigvee_{\alpha\in A}\chi^U_{RL}(q_\alpha)$.

3. $\chi^U_{RL}(1) = \mathfrak{e}$.

4. $\neg_B \chi^U_{RL}(p) = \bigvee\{r \in \chi^U_{RL}(B_U) \mid \chi^U_{RL}(p) \wedge r = 0\} = \chi^U_{RL}(p^\perp)$.

5. $\chi^U_{RL} : (B_U, \leq, \bigvee, 1, 0) \cong (\chi^U_{RL}(B_U), \leq, \bigvee, \mathfrak{e}, 0)$.

6. $\chi^U_{RL}(p)\&\chi^U_{RL}(q) = \chi^U_{RL}(p) \wedge \chi^U_{RL}(q) = \chi^U_{RL}(p \wedge q)$.

7. $\chi^U_{RL}(p)^* = \chi^U_{RL}(p)$.

Proof. Let $q_j = \sigma_U(p_j)$ for each $j \in J$.

1. If $p = \bigvee_{j\in K} q_j$ and $q = \bigvee_{k \in K'} q_k$, where $K, K' \subset J$, then $p \leq q \iff K \subset K'$. Hence

$$p \leq q \implies \chi^U_{RL}(p) = \bigvee_{j\in K} \chi^U_{RL}(q_j) \leq \bigvee_{j \in K'} \chi^U_{RL}(q_j) = \chi^U_{RL}(q)$$

$$\chi^U_{RL}(p) \leq \chi^U_{RL}(q) \implies p = (\chi^U_{RL}(p))(1) \leq (\chi^U_{RL}(q))(1) = q.$$

2. If $q_\alpha = \bigvee_{j \in K_\alpha} q_j$ for each $\alpha \in A$, where $K_\alpha \subset J$,

$$\chi^U_{RL}(\textstyle\bigvee_{\alpha\in A} q_\alpha) = \chi^U_{RL}(\bigvee_{j \in \bigcup_{\alpha\in A} K_\alpha} q_j) = \bigvee_{j\in\bigcup_{\alpha\in A} K_\alpha} \chi^U_{RL}(q_j) = \bigvee_{\alpha\in A} \chi^U_{RL}(q_\alpha).$$

3. $(\chi^U_{RL}(q_j))(q_k) = \kappa_{q_j}(\lambda_{q_j^\perp}(q_k)) = \begin{cases} 0 & k \neq j \\ q_k & k = j \end{cases}$.

$$\therefore \quad (\chi^U_{RL}(1))(q_k) = (\bigvee_{j\in J} \chi^U_{RL}(q_j))(q_k) = \bigvee_{j \in J} (\chi^U_{RL}(q_j))(q_k) = q_k = \mathfrak{e}(q_k).$$

$$\therefore \quad (\chi^U_{RL}(1))(q) = (\chi^U_{RL}(1))(\bigvee_{k\in K} q_k) = \bigvee_{k\in K} (\chi^U_{RL}(1))(q_k) = \bigvee_{k \in K} q_k = q.$$

4. If $p = \bigvee_{j\in K} \sigma_U(p_j)$, then $p^\perp = \bigvee_{j \in J-K} \sigma_U(p_j)$. Hence,

$$\begin{aligned} \neg_B \chi^U_{RL}(p) &= \bigvee\{r \in \chi^U_{RL}(B_U) \mid \chi^U_{RL}(p) \wedge r = 0\} \\ &= \bigvee_{j\in J-K} \chi^U_{RL}(\sigma_U(p_j)) = \chi^U_{RL}(p^\perp). \end{aligned}$$

5. By 1, 2, 3.

6. By Lemma 3.20,

$$\begin{aligned} \chi^U_{RL}(p) \,\&\, \chi^U_{RL}(q) &= \bigvee_{j\in K} \bigvee_{k \in K'} \chi_{RL}(q_j) \,\&\, \chi_R(q_k)\&\chi_L(q_k) \\ &= \bigvee_{j \in K\cap K'} \chi_R(q_j)\&\chi_L(q_j) = \chi^U_{RL}(p \wedge q). \end{aligned}$$

7. is obvious.

□

Corollary 3.22. *1. For each unitary operator* $U \in \mathcal{U}$,
$B = B_E \cong B_U \cong \chi_L(B_U) \cong \chi_R(B_U) \cong \chi^U_{RL}(B_U) \cong \mathcal{P}(J)$ *(Theorem 3.21).*

2. Sub-quantales $R(\mathcal{Q})$ *and* $L(\mathcal{Q})$ *of* $\mathcal{Q}$ *are mutually conjugate.*

$$a \in R(\mathcal{Q}) \iff a^* \in L(\mathcal{Q})$$

Definition 3.23.

$$\begin{aligned}
\mathrm{Sh}_{\mathcal{U}}(\chi_L(B)) &\stackrel{\text{def}}{=} \left(\widetilde{\chi_L(B)}, \pi\right) \\
\text{where} \quad & \widetilde{\chi_L(B)} \stackrel{\text{def}}{=} \{< U, \chi^U_L(\sigma_U(p)) >\mid U \in \mathcal{U},\ p \in B\} \\
& \pi\left(< U, \chi^U_L(\sigma_U(p)) >\right) \stackrel{\text{def}}{=} U \quad \text{for } U \in \mathcal{U},\ p \in B \\
\mathrm{Sh}_{\mathcal{U}}(\chi_R(B)) &\stackrel{\text{def}}{=} \left(\widetilde{\chi_R(B)}, \pi\right) \\
\text{where} \quad & \widetilde{\chi_R(B)} \stackrel{\text{def}}{=} \{< U, \chi^U_R(\sigma_U(p)) >\mid U \in \mathcal{U},\ p \in B\} \\
& \pi\left(< U, \chi^U_R(\sigma_U(p)) >\right) \stackrel{\text{def}}{=} U \quad \text{for } U \in \mathcal{U},\ p \in B \\
\mathrm{Sh}_{\mathcal{U}}(\chi_{RL}(B)) &\stackrel{\text{def}}{=} \left(\widetilde{\chi_{RL}(B)}, \pi\right) \\
\text{where} \quad & \widetilde{\chi_{RL}(B)} \stackrel{\text{def}}{=} \{< U, \chi^U_{RL}(\sigma_U(p)) >\mid U \in \mathcal{U},\ p \in B\} \\
& \pi\left(< U, \chi^U_{RL}(\sigma_U(p)) >\right) \stackrel{\text{def}}{=} U \quad \text{for } U \in \mathcal{U},\ p \in B
\end{aligned}$$

By Corollary 3.22, $\mathrm{Sh}_{\mathcal{U}}(\chi_R(B))$, $\mathrm{Sh}_{\mathcal{U}}(\chi_L(B))$, $\mathrm{Sh}_{\mathcal{U}}(\chi_{RL}(B))$ are sheaves of complete Boolean lattice isomorphic to $\mathrm{Sh}_{\mathcal{U}}(B)$.

Theorem 3.24. *Let* $U \in \mathcal{U}$, *and* $q_j = \sigma_U(p_j)$ *for* $j \in J$.

1. $p \le q \Longrightarrow \chi^U_{LR}(p) \le \chi^U_{LR}(q)$, *for* $p, q \in B_U$.
2. $\chi^U_{LR}(\bigvee_{\alpha \in A} q_\alpha) = \bigvee_{\alpha \in A} \chi^U_{LR}(q_\alpha)$, *for* $\{q_\alpha\}_{\alpha \in A} \subset B_U$.
3. $\chi^U_{LR}(q_j) = 1$ *for* $q_j \in P_U$, *hence* $\chi^U_{LR}(1) = 1$ *on* B_U.
4. $\neg_{\mathbf{2}}\chi^U_{LR}(p) = \bigvee\{r \in \mathbf{2} \mid \chi^U_{LR}(p) \wedge r = 0\} = \chi^U_{LR}(\neg_{\mathbf{2}} p)$, *for* $p \in B_U$.
5. *Mapping* $\chi^U_{LR} : B_U \to \mathbf{2}$ *preserves* $\bigvee$ *and* $\neg_{\mathbf{2}}$.
6. $\chi^U_{LR}(p) \& \chi^U_{LR}(q) = \chi^U_{LR}(p) \wedge \chi^U_{LR}(q)$, *for* $p, q \in B_U$.

Proof. 1. If $p = \bigvee_{j \in K} q_j$, $q = \bigvee_{k \in K'} q_k$ and $p \le q$, then $K \subset K'$. Therefore, $\chi^U_{LR}(p) \le \chi^U_{LR}(q)$.

2. Since $\chi^U_{LR}(p)$ is 1 or 0 for all $p \in B_U$, it suffices to show that

$$\chi^U_{LR}(\bigvee_{\alpha\in A} q_\alpha) = 1 \Longleftrightarrow \bigvee_{\alpha\in A} \chi^U_{LR}(q_\alpha) = 1.$$

If $q_\alpha = \bigvee_{j\in K_\alpha} q_j$ for each $\alpha \in A$,

$$\chi^U_{LR}(\bigvee_{\alpha\in A} q_\alpha) = 1 \Longleftrightarrow \bigcup_{\alpha\in A} K_\alpha \neq \emptyset \Longleftrightarrow \bigvee_{\alpha\in A} \chi^U_{LR}(q_\alpha) = 1.$$

3. By Theorem 3.18, $\chi_L(q_j)\&\chi_R(q_k) = \kappa_{q_j}\&\lambda_{q_k^\perp} = \begin{cases} 0 & k \neq j, \\ 1 & k = j. \end{cases}$

Hence, $\chi^U_{LR}(1) \geq \chi^U_{LR}(q_j) = \chi_L(q_j)\&\chi_R(q_j) = 1$.

4. Let $p = \bigvee_{j\in K} \sigma_U(p_j)$. Then $p^\perp = \bigvee_{j\in J-K} \sigma_U(p_j)$. Hence,

$$\neg_2\chi^U_{LR}(p) = \begin{cases} 0 & p \neq 0, \\ 1 & p = 0 \end{cases} = \begin{cases} 0 & \neg_2 p = 0, \\ 1 & \neg_2 p = 1 \end{cases} = \chi^U_{LR}(\neg_2 p)$$

5. By 1,2,3.

6. Since $\chi^U_{LR}(p)$ and $\chi^U_{LR}(q)$ are 1 or 0, $\chi^U_{LR}(p)\,\&\chi^U_{LR}(q) = 1$ or 0, and

$$\begin{aligned} \chi^U_{LR}(p)\,\&\chi^U_{LR}(q) = 1 &\Longleftrightarrow \chi^U_{LR}(p) = 1 \text{ and } \chi^U_{LR}(q) = 1 \\ &\Longleftrightarrow \chi^U_{LR}(p) \wedge \chi^U_{LR}(q) = 1. \end{aligned}$$

Therefore, $\chi^U_{LR}(p)\,\&\chi^U_{LR}(q) = \chi^U_{LR}(p) \wedge \chi^U_{LR}(q)$.

□

3.4 Quantale valued logic QtL

Quantale valued logic QtL is obtained from lattice valued logic by adding new propositional symbol $\mathfrak{e}$ and logical operators $\&$, $^\perp$, * together with the following logical axioms QA1 – QA7, where formula $\overline{\varphi}$ with the overline is a global formula, i.e. $\vdash \overline{\varphi} \Leftrightarrow \Box_2\overline{\varphi}$, and

$$\text{Right}(\varphi) \stackrel{\text{def}}{\Longleftrightarrow} (\varphi\&\top \rightarrow_2 \varphi); \quad \text{Left}(\varphi) \stackrel{\text{def}}{\Longleftrightarrow} (\top\&\varphi \rightarrow_2 \varphi).$$

QA1 $[(\varphi \rightarrow_2 \varphi') \wedge (\psi \rightarrow_2 \psi')] \Rightarrow [(\varphi\&\psi) \rightarrow_2 (\varphi'\&\psi')]$

QA2 $\overline{\varphi}\,\&\,\overline{\psi} \Longleftrightarrow \overline{\varphi} \wedge \overline{\psi}$

QA3 (Distributivity)

1. $\varphi\&(\psi \vee \xi) \Leftrightarrow (\varphi\&\psi) \vee (\varphi\&\xi)$
2. $(\varphi \vee \psi)\&\xi \Leftrightarrow (\varphi\&\xi) \vee (\psi\&\xi)$
3. $\varphi\&\exists x\psi(x) \Leftrightarrow \exists x(\varphi\&\psi(x))$, where φ has no occurrance of x
4. $\exists x\varphi(x)\,\&\,\psi \Leftrightarrow \exists x\big(\varphi(x)\&\psi\big)$, where ψ has no occurrance of x

QA4 (Unitarity) $\mathfrak{e}\&\varphi \Leftrightarrow \varphi\&\mathfrak{e} \Leftrightarrow \varphi$

QA5 (Involutivity)

1. $\overline{\varphi} \Leftrightarrow (\overline{\varphi})^*$
2. $(\varphi \rightarrow_2 \psi) \Leftrightarrow (\varphi^* \rightarrow_2 \psi^*)$
3. $\varphi^{**} \Leftrightarrow \varphi$
4. $(\varphi\&\psi)^* \Leftrightarrow \psi^*\&\varphi^*$
5. $(\varphi \vee \psi)^* \Leftrightarrow (\varphi^* \vee \psi^*)$
6. $(\exists x\varphi(x))^* \Leftrightarrow \exists x\varphi^*(x)$

QA6 (Gelfand Property) $\mathrm{Right}(\varphi) \Rightarrow [(\varphi\&\varphi^*\&\varphi) \leftrightarrow_2 \varphi]$

QA7 (Orthomodularity)

1. $\mathrm{Right}(\varphi), \mathrm{Right}(\psi) \Rightarrow [(\varphi^*\&\psi \rightarrow_2 \bot) \leftrightarrow (\psi \rightarrow_2 \varphi^\bot)]$
2. $\mathrm{Right}(\varphi), \varphi^{\bot\bot} \Rightarrow \varphi$
3. $\mathrm{Right}(\varphi), \mathrm{Right}(\psi), (\varphi \rightarrow_2 \psi), \psi \Rightarrow (\psi \wedge \varphi) \vee (\psi \wedge \varphi^\bot)$

Theorem 3.25. *In quantale valued logic* QtL*, the following sequents are provable, where $\overline{\varphi}$ is a global formula.*

1. $\Rightarrow \mathrm{Right}(\overline{\varphi})$ *and* $\Rightarrow \mathrm{Left}(\overline{\varphi})$
2. $\mathrm{Right}(\varphi) \Leftrightarrow \mathrm{Left}(\varphi^*)$
3. $\mathrm{Right}(\varphi) \Rightarrow (\varphi\&\varphi \leftrightarrow_2 \varphi)$
4. $\mathrm{Right}(\varphi), \varphi \Rightarrow \varphi^*\&\varphi$
5. $\mathrm{Right}(\varphi), \mathrm{Right}(\psi), \varphi \wedge \psi \Rightarrow \varphi^*\&\psi$
6. $\mathrm{Right}(\varphi), \mathrm{Right}(\psi) \Rightarrow [(\varphi \vee \psi)^\bot \leftrightarrow_2 (\varphi^\bot \wedge \psi^\bot)]$
7. $\mathrm{Right}(\varphi), \varphi \wedge \varphi^\bot \Rightarrow$
8. $\overline{\varphi}^\bot \Leftrightarrow \neg_2\overline{\varphi}$
9. $\varphi \Leftrightarrow \varphi^{\bot\bot}$.
10. $(\varphi \rightarrow_2 \psi) \Rightarrow (\psi^\bot \rightarrow_2 \varphi^\bot)$
11. $(\varphi \rightarrow_2 \psi)\&\xi \Rightarrow (\varphi\&\xi \rightarrow_2 \psi\&\xi)$, $\xi\&(\varphi \rightarrow_2 \psi) \Rightarrow (\xi\&\varphi \rightarrow_2 \xi\&\psi)$

Proof. 1. By QA 2.

2. $\vdash \mathrm{Right}(\varphi) \Leftrightarrow (\varphi\&\top \rightarrow_2 \varphi) \Leftrightarrow (\top\&\varphi^* \rightarrow_2 \varphi^*) \Leftrightarrow \mathrm{Left}(\varphi^*)$ by QA5.

3. $\vdash \text{Right}(\varphi) \Rightarrow [(\varphi\&\varphi) \to_2 \varphi]$ is trivial.

$$\begin{aligned} & \vdash \text{Right}(\varphi) && \Rightarrow && [(\varphi\&\varphi^*) \to_2 \varphi] \\ \therefore\ & \vdash \text{Right}(\varphi) && \Rightarrow && ((\varphi\&\varphi^*)\&\varphi) \to_2 \varphi\&\varphi \quad \text{by QA3 (5)} \\ \therefore\ & \vdash \text{Right}(\varphi) && \Rightarrow && \varphi \to_2 (\varphi\&\varphi) \quad \text{by Gelfand property.} \end{aligned}$$

4. $\vdash \text{Right}(\varphi) \Rightarrow \text{Left}(\varphi^*\&\varphi)$. Hence, $\vdash \text{Right}(\varphi) \Rightarrow (\varphi\&\varphi^*\&\varphi \to_2 \varphi^*\&\varphi)$.
   $\therefore \vdash \text{Right}(\varphi) \Rightarrow (\varphi \to_2 \varphi^*\&\varphi)$ by Gelfand property.

5. $\vdash (\varphi \wedge \psi)\&\top \Rightarrow (\varphi\&\top) \wedge (\psi\&\top) \Rightarrow (\varphi \wedge \psi)$ by QA1.
   $\therefore \vdash \text{Right}(\varphi), \text{Right}(\psi) \Rightarrow \text{Right}(\varphi \wedge \psi)$,
   $\vdash \text{Right}(\varphi \wedge \psi) \Rightarrow (\varphi \wedge \psi) \to_2 (\varphi \wedge \psi)^*\&(\varphi \wedge \psi)$ by (4),
   and $\vdash (\varphi \wedge \psi)^*\&(\varphi \wedge \psi) \Rightarrow \varphi^*\&\psi$.
   $\therefore \vdash \text{Right}(\varphi), \text{Right}(\psi) \Rightarrow (\varphi \wedge \psi) \to_2 (\varphi^*\&\psi)$.

6. $\vdash \text{Right}(\varphi), \text{Right}(\psi),\ (\varphi \vee \psi)\&\top \Rightarrow (\varphi\&\top) \vee (\psi\&\top) \Rightarrow (\varphi \vee \psi)$ by QA3.
   $\therefore \vdash \text{Right}(\varphi), \text{Right}(\psi) \Rightarrow \text{Right}(\varphi \vee \psi)$

   $\vdash \text{Right}(\varphi \vee \psi) \Rightarrow [(\varphi \vee \psi)^*\&(\varphi \vee \psi)^\perp \to_2 \bot]$ by QA7.
   $\Rightarrow [\varphi^*\&(\varphi \vee \psi)^\perp \to_2 \bot] \wedge [\psi^*\&(\varphi \vee \psi)^\perp \to_2 \bot]$
   $\Rightarrow [(\varphi \vee \psi)^\perp \to_2 \varphi^\perp] \wedge [(\varphi \vee \psi)^\perp \to_2 \psi^\perp]$
   $\Rightarrow [(\varphi \vee \psi)^\perp \to_2 (\varphi^\perp \wedge \psi^\perp)]$

   $\vdash \text{Right}(\varphi), \text{Right}(\psi), \text{Right}(\xi), (\xi \to_2 \varphi^\perp \wedge \psi^\perp)$
   $\Rightarrow (\xi \to_2 \varphi^\perp) \wedge (\xi \to_2 \psi^\perp)$
   $\Rightarrow [(\varphi^*\&\xi) \vee (\psi^*\&\xi) \to_2 \bot]$
   $\Rightarrow [(\varphi \vee \psi)^*\&\xi) \to_2 \bot]$
   $\Rightarrow [\xi \to_2 (\varphi \vee \psi)^\perp]$

7. $\vdash \text{Right}(\varphi), \varphi \wedge \varphi^\perp \Rightarrow \varphi^*\&\varphi^\perp \Rightarrow \bot$.

8. $\vdash \text{Right}(\varphi), \overline{\varphi}^*\&\neg\overline{\varphi} \Rightarrow \overline{\varphi} \wedge \neg\overline{\varphi} \Rightarrow \bot$, by QA2 and QA5.
   $\therefore \vdash \text{Right}(\varphi) \Rightarrow (\overline{\varphi}^\perp \leftrightarrow \neg\overline{\varphi})$, by QA7.

9. $\vdash \text{Right}(\varphi), \varphi^*\&\varphi^\perp \Rightarrow \bot \quad \therefore \vdash \text{Right}(\varphi), (\varphi^\perp)^*\&\varphi \Rightarrow \bot$.
   $\therefore \vdash \text{Right}(\varphi), \varphi \Rightarrow \varphi^{\perp\perp}$.

10. $\vdash \text{Right}(\varphi), \text{Right}(\psi), (\varphi \to_2 \psi) \wedge (\varphi^*\&\psi^\perp) \Rightarrow \psi^*\&\psi^\perp \Rightarrow \bot$, by QA5.
    $\therefore \vdash \text{Right}(\varphi), \text{Right}(\psi), (\varphi \to_2 \psi) \Rightarrow (\psi^\perp \to_2 \varphi^\perp)$.

11. Since $\vdash (\varphi \to_2 \psi)\&\xi \Rightarrow (\varphi \to_2 \psi)\&(\xi \to_2 \xi)$,

$$\vdash (\varphi \to_2 \psi)\&\xi \Rightarrow (\varphi\&\xi \to_2 \psi\&\xi) \text{ by QA1.}$$

    Similarly, $\vdash \xi\&(\varphi \to_2 \psi) \Longrightarrow (\xi\&\varphi \to_2 \xi\&\psi)$.

$\square$

Hilbert quantale (Definition 3.15) is a complete lattice with &, $*$, $\mathfrak{e}$ which satisfies conditions corresponding to axioms QA1–QA7 by Theorem 3.12. Hence, we have

Theorem 3.26. *Hilbert quantale is a model of* QtL.

3.5 Quantale valued set theory QtZFZ

Quantale valued set theory QtZFZ (Quantale valued Zermero-Fraenkel set theory with Zorn's Lemma) is obtained by adjoining quantale valued logic QtL and nonlogical axioms A1–A11 of lattice valued set theory LZFZ (Theorem 2.30). That is, QtZFZ is a lattice valued set theory with additional logical operator &, $^{\perp}$, *, constant term $\mathfrak{e}$ and logical axioms QA1–QA7.

It is obvious that $V^{\mathcal{L}}$ is a model of QtZFZ, if $\mathcal{L}$ is a Hilbert quantale.

3.6 Quantale valued universe $V^{\mathcal{Q}}$

Quantale valued universe $V^{\mathcal{Q}}$, where $\mathcal{Q} =< Q(P(\mathcal{H})), \leq, \bigvee, \&, *, \mathfrak{e} >$ is a quantale of endomorphisms of $P(\mathcal{H})$ (Definition 3.15), is a lattice valued universe:

$$\begin{aligned} V^{\mathcal{Q}}_{\alpha} &= \{u \mid \exists\beta < \alpha\, \exists\mathcal{D}u \subset V^{\mathcal{Q}}_{\beta}(u : \mathcal{D}u \to \mathcal{Q})\}, \\ V^{\mathcal{Q}} &= \textstyle\bigcup_{\alpha\in On} V^{\mathcal{Q}}_{\alpha}, \end{aligned}$$

where V is an external universe which is a model of classical set theory ZFC.

For $u, v \in V^{\mathcal{Q}}$, $[\![u =_{\mathbf{2}} v]\!]_{\mathcal{Q}}$ and $[\![u \in_{\mathbf{2}} v]\!]_{\mathcal{Q}}$ are defined by induction on the rank of u, v.

$$\begin{aligned} [\![u =_{\mathbf{2}} v]\!]_{\mathcal{Q}} &= \bigwedge_{x\in\mathcal{D}u} (u(x) \to_{\mathbf{2}} [\![x \in_{\mathbf{2}} v]\!]_{\mathcal{Q}}) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \to_{\mathbf{2}} [\![x \in_{\mathbf{2}} u]\!]_{\mathcal{Q}}) \\ [\![u \in_{\mathbf{2}} v]\!]_{\mathcal{Q}} &= \bigvee_{x\in\mathcal{D}v} [\![u =_{\mathbf{2}} x]\!]_{\mathcal{Q}} \wedge v(x). \end{aligned}$$

Primitive logical symbols $\wedge$, $\vee$, &, $^{\perp}$, *, $\to_{\mathbf{2}}$, $\neg_{\mathbf{2}}$, $\forall$, and $\exists$ of the quantale valued logic are interpreted in $V^{\mathcal{Q}}$ as the corresponding operators on $\mathcal{Q}$. Nonlogical axioms A1–A11 of lattice valued set theory are valid in $V^{\mathcal{Q}}$.

3.7 Sheaves of Boolean valued universe in $V^{\mathcal{Q}}$

By Theorem 2.35, we have

Theorem 3.27. $(\bigvee, ^{\perp})$*-preserving bijections* χ_R, χ_L, χ^U_{RL} *and* $*$ *induce bijections of universes, and also bijections of sheaves of universes:*

1. $\chi_R : P(\mathcal{H}) \cong R(\mathcal{Q})$, $\quad V^{P(\mathcal{H})} \cong V^{R(\mathcal{Q})}$, $\quad \mathrm{Sh}_{\mathcal{U}}(V^B) \cong \mathrm{Sh}_{\mathcal{U}}(\chi_R(V^B))$,

$$\textit{where} \quad \mathrm{Sh}_{\mathcal{U}}(\chi_R(V^B)) =< \widetilde{\chi_R(V^B)}, \pi >,$$

$$\widetilde{\chi_R(V^B)} = \{< U, \chi_R(\sigma_U(t)) >\mid U \in \mathcal{U},\ t \in V^B\}, \quad \pi :< U, \chi_R(\sigma_U(t)) > \mapsto U,$$

2. $\chi_L : P(\mathcal{H}) \cong L(\mathcal{Q}), \quad V^{P(\mathcal{H})} \cong V^{L(\mathcal{Q})}, \quad \mathrm{Sh}_{\mathcal{U}}(V^B) \cong \mathrm{Sh}_{\mathcal{U}}(\chi_L(V^B)),$

$$\textit{where} \quad \mathrm{Sh}_{\mathcal{U}}(\chi_L(V^B)) = <\widetilde{\chi_L(V^B)}, \pi>,$$

$$\widetilde{\chi_L(V^B)} = \{<U, \chi_L(\sigma_U(t))>| \, U \in \mathcal{U}, \ t \in V^B\}, \quad \pi :< U, \chi_L(\sigma_U(t)) > \mapsto U,$$

3. $\chi^U_{RL} :< B, \bigvee, ^\perp, 1 > \cong < \chi^U_{RL}(B_U), \bigvee, \neg_{\mathbf{2}}, \mathfrak{e} >, \quad V^B \cong V^{\chi^U_{RL}(B_U)}.$

$$\mathrm{Sh}_{\mathcal{U}}(\chi_{RL}(V^B)) \stackrel{def}{=} < \widetilde{\chi_{RL}(V^B)}, \pi >, \quad \textit{where}$$

$$\widetilde{\chi_{RL}(V^B)} \stackrel{def}{=} \{< U, \chi^U_{RL}(\sigma_U(t)) >| \, U \in \mathcal{U}, \ t \in V^B\}, \quad \pi :< U, \chi^U_{RL}(\sigma(t)) > \mapsto U,$$

$$\textit{Then} \quad \mathrm{Sh}_{\mathcal{U}}(V^B) \cong \mathrm{Sh}_{\mathcal{U}}(\chi_{RL}(V^B)).$$

4. $* :< R(\mathcal{Q}), \bigvee, ^\perp > \cong < L(\mathcal{Q}), \bigvee, ^{*\perp *} >, \quad V^{R(\mathcal{Q})} \cong V^{L(\mathcal{Q})}, \ \textit{hence}$

$$\mathrm{Sh}_{\mathcal{U}}(\chi_R(V^B) \cong \mathrm{Sh}_{\mathcal{U}}(\chi_L(V^B)).$$

$V^{\chi_R(B_U)}$, $V^{\chi_L(B_U)}$ and $V^{\chi^U_{RL}(B_U)}$ are Boolean valued universe of classical set theory for each $U \in \mathcal{U}$.

4 A completeness of QtZFZ

In this section we will prove that a universe of QtZFZ is represented in QtZFZ itself (Section 4.7), and QtZFZ is complete in the following sense :

For a sentence φ of QtZFZ,

$$\text{ZFC} \quad \vdash \quad \text{“ } [\![\varphi]\!] = 1 \text{ in } V^Q \text{ for all Hilbert quantale } Q \text{ ”}$$
$$\Longrightarrow \quad \text{QtZFZ} \vdash \varphi.$$

We will outline the proof of the completeness of QtZFZ, which is similar to the proof of the completeness of lattice valued set theory LZFZ in [12].

Theorems of LZFZ is also a theorem of QtZFZ.

4.1 Recursion Principle

Binary relation $\prec$ is said to be **well-founded**, if

WF1 $\forall x, y \neg (x \prec y \wedge y \prec x)$

WF2 $\forall x \in_{\mathbf{2}} \mathrm{Fld}(\prec)[\forall y(y \prec x \to_{\mathbf{2}} \varphi(y)) \to_{\mathbf{2}} \varphi(x)] \to_{\mathbf{2}} \forall x \in_{\mathbf{2}} \mathrm{Fld}(\prec)(\varphi(x)),$

where $\mathrm{Fld}(\prec) \stackrel{\text{def}}{=} \{x \mid \exists y(x \prec y) \vee \exists y(y \prec x)\}.$

WF3 $\forall x \exists y \forall z (z \prec x \to_{\mathbf{2}} z \in_{\mathbf{2}} y)$

In view of the axiom A9 ($\in_{\mathbf{2}}$-induction), the membership relation $\in_{\mathbf{2}}$ is a well-founded relation, and so is $\in_{\mathbf{2}}^{\square}$, where

$$x \in_{\mathbf{2}}^{\square} y \overset{\text{def}}{\Longleftrightarrow} \square_{\mathbf{2}}(x \in_{\mathbf{2}} y).$$

Singlton $\{x\}$ and ordered pair $< x, y >$ are defined as usual:

$$\{x\} \overset{\text{def}}{=} \{x, x\}, \quad < x, y > \overset{\text{def}}{=} \{\{x\}, \{x, y\}\}$$

Then

$$\vdash x \in_{\mathbf{2}} \{y\} \;\Leftrightarrow\; x =_{\mathbf{2}} y, \qquad \vdash < x, y > =_{\mathbf{2}} < x', y' > \Leftrightarrow\; x =_{\mathbf{2}} x' \wedge y =_{\mathbf{2}} y'.$$

Definition 4.1. A binary relation $F(x, y)$ is said to be **global**, if

$$\forall x, y\, [F(x, y) \to_{\mathbf{2}} \square_{\mathbf{2}} F(x, y)]\,;$$

and a global relation $F(x, y)$ is **functional**, if

$$\forall x, y, y'\, [F(x, y) \wedge F(x, y') \to_{\mathbf{2}} y =_{\mathbf{2}} y']\,.$$

For a global functional relation F, we write $F(x) =_{\mathbf{2}} y$ instead for $F(x, y)$. If F is a global functional relation and $\prec$ is a well-founded relation, then $F_{\prec u}$ is defined for each set $u \in \mathrm{Fld}(\prec)$ by

$$F_{\prec u} \overset{\text{def}}{=} \{< x, y > \mid F(x, y) \wedge \Diamond_{\mathbf{2}}(x \prec u)\,\}.$$

$F_{\prec u}$ is a global set of LZFZ, i.e. $\vdash \exists x(F_{\prec u} \in_{\mathbf{2}} x) \wedge \mathrm{Gl}(F_{\prec u})$, by WF3, A11($\Diamond_{\mathbf{2}}$) and A8(Collection).

The following recursion principle is justified in LZFZ (cf. [12]), and so is in QtZFZ.

Theorem 4.2 (Recursion Principle)**.** *Let $\prec$ be a well founded relation and $H(a, b)$ be a global functional relation such that $\forall x \exists y H(x, y)$. Then there exists a unique global functional relation F such that*

$$\mathrm{Dom}\, F = \mathrm{Fld}(\prec) \;\wedge\; \forall x \in \mathrm{Fld}(\prec)\, (F(x) = H(F_{\prec x}))\,.$$

4.2 Check sets

For any complete lattice $\mathcal{L}$, set $\{1, 0\}$, where $1 = \bigvee \mathcal{L}$, $0 = \bigwedge \mathcal{L}$, is a Boolean sublattice, and denoted by $\mathbf{2}$. Let V be a universe of ZFC in which $V^{\mathcal{L}}$ is constructed. V is embedded into $V^{\mathcal{L}}$ as sub-universe $V^{\mathbf{2}}$:

$$u \mapsto \check{u} \in V^{\mathcal{L}} \quad \text{for } u \in V,$$

where $\check{u}$ is defined inductively by:

$$\begin{cases} \mathcal{D}\check{u} = \{\check{x} \mid x \in u\}, \\ \check{u}(\check{x}) = 1. \end{cases}$$

For sets $u, v \in V$, $[\![\check{u} =_{\mathbf{2}} \check{v}]\!]_{\mathcal{L}}$ and $[\![\check{u} \in_{\mathbf{2}} \check{v}]\!]_{\mathcal{L}}$ are global, i.e. $\in \mathbf{2}$, and

Lemma 4.3. *For $u, v \in V$,*

$$u = v \Longleftrightarrow [\![\check{u} =_{\mathbf{2}} \check{v}]\!]_{\mathcal{L}} = 1 \quad and \quad u \in v \Longleftrightarrow [\![\check{u} \in_{\mathbf{2}} \check{v}]\!]_{\mathcal{L}} = 1.$$

Furthermore, we have

Theorem 4.4 (Titani [12]). *If $\varphi(u_1, \cdots u_n)$ is a bounded sentence of* ZFC *with constants $u_1, \cdots u_n$ in V, then $[\![\varphi^{(2)}(\check{u}_1, \cdots \check{u}_n)]\!]_{\mathcal{L}}$ is global, and*

$$\varphi(u_1, \cdots u_n) \Longleftrightarrow [\![\varphi^{(2)}(\check{u}_1, \cdots \check{u}_n)]\!]_{\mathcal{L}} = 1,$$

where the sentence of ZFC *means a sentence constructed from atomic formulas of the forms $u = v$, $u \in v$ by using logical operator $\vee$, $\wedge$, $\neg$, $\forall$, $\exists$; and $\varphi^{(2)}$ is the formula obtained from φ by replacing $\in$, $=$,$\neg$ by $\in_{\mathbf{2}}$, $=_{\mathbf{2}}$, $\neg_{\mathbf{2}}$.*

Definition 4.5. Element of the sub-universe $V^{\mathbf{2}}$ of lattice valued universe $V^{\mathcal{L}}$ is called a **check set**.

The notion of 'check set' is defined in QtZFZ by $\in_{\mathbf{2}}^{\Box}$-recursion:

$$\mathrm{ck}(x) \overset{\text{def}}{\Longleftrightarrow} \forall t \, (t \in_{\mathbf{2}} x \;\leftrightarrow_{\mathbf{2}}\; t \in_{\mathbf{2}}^{\Box} x \wedge \mathrm{ck}(t)) \, .$$

Theorem 4.6. *For an arbitrary complete lattice $\mathcal{L}$ and $x \in V^{\mathcal{L}}$,*

$$[\![\, \mathrm{ck}(x) \,]\!]_{\mathcal{L}} = 1 \Longrightarrow \textit{ there exists a check set } \check{t} \textit{ for } t \in V \textit{ such that } [\![x =_{\mathbf{2}} \check{t} \,]\!] = 1.$$

The **class of check sets** will be denoted by W:

$$x \in_{\mathbf{2}} W \overset{\text{def}}{\Longleftrightarrow} \mathrm{ck}(x).$$

4.3 Ordinals

Definition 4.7. Formula $\mathrm{Ord}(\alpha)$ ("α is an ordinal") is defined by $\in_{\mathbf{2}}$-recursion:

$$\begin{aligned}
\mathrm{Ord}(\alpha) \quad &\overset{\text{def}}{\Longleftrightarrow} \quad \mathrm{Gl}(\alpha) \wedge \mathrm{Tr}(\alpha) \wedge \forall \beta \in_{\mathbf{2}} \alpha \, [\mathrm{Gl}(\beta) \wedge \mathrm{Tr}(\beta) \,] \, , \quad \text{where} \\
&\qquad \mathrm{Gl}(\alpha) \overset{\text{def}}{\Longleftrightarrow} \forall \beta (\beta \in_{\mathbf{2}} \alpha \rightarrow_{\mathbf{2}} \beta \in_{\mathbf{2}}^{\Box} \alpha) \\
&\qquad \mathrm{Tr}(\alpha) \overset{\text{def}}{\Longleftrightarrow} \forall \beta, \gamma (\beta \in_{\mathbf{2}} \alpha \wedge \gamma \in_{\mathbf{2}} \beta \;\rightarrow_{\mathbf{2}}\; \gamma \in_{\mathbf{2}} \alpha) \\
\mathrm{On} \quad &\overset{\text{def}}{=} \quad \{\alpha \mid \mathrm{Ord}(\alpha)\}
\end{aligned}$$

Definition 4.8. A global well founded relation $\prec$ is called a **well order on** u if

$$(\mathrm{Fld}(\prec) =_{\mathbf{2}} u) \wedge (\prec \text{ is transitive}) \wedge (\prec \text{ is extensional}), \quad \text{where}$$

$$\prec \text{ is transitive} \overset{\text{def}}{\Longleftrightarrow} \forall x, y, z [(x \prec y) \wedge (y \prec z) \rightarrow_{\mathbf{2}} (x \prec z)]$$

$$\prec \text{ is extensional} \overset{\text{def}}{\Longleftrightarrow} \forall x, y [x, y \in_{\mathbf{2}} u \wedge \forall z (z \prec x \leftrightarrow_{\mathbf{2}} z \prec y) \rightarrow_{\mathbf{2}} x =_{\mathbf{2}} y].$$

The following theorems are provable in LZFZ (cf. [12]), and so is in QtZFZ.

Theorem 4.9. *Every global set can be well-ordered. That is, for every global set u, there exists a well-order $\prec$ on u and ordinal α such that $< u, \prec >$ is order isomorphic to an ordinal $< \alpha, \in_{\mathbf{2}} >$, i.e. there exists ρ such that*

$$(\rho : u \to \alpha) \wedge (\rho(u) =_{\mathbf{2}} \alpha) \wedge \forall x, y \in_{\mathbf{2}} u[x \prec y \leftrightarrow_{\mathbf{2}} \rho(x) \in_{\mathbf{2}} \rho(y)] \wedge$$
$$\forall x, y \in u[x =_{\mathbf{2}} y \leftrightarrow_{\mathbf{2}} \rho(x) =_{\mathbf{2}} \rho(y)].$$

4.4 Model W of ZFC in QtZFZ

An interpretation of ZFC in QtZFZ is obtained by relativizing the range of quantifiers to check sets. That is, let

$$W \stackrel{\text{def}}{=} \{x \mid \text{ck}(x)\}, \quad \text{where } \text{ck}(x) \stackrel{\text{def}}{\Longleftrightarrow} \forall t[t \in_{\mathbf{2}} x \leftrightarrow_{\mathbf{2}} (t \in_{\mathbf{2}}^{\square} x \wedge \text{ck}(t))].$$

Quantifiers relativized on check sets are denoted by $\forall^W, \exists^W$, :

$$\forall^W x\varphi(x) \stackrel{\text{def}}{\Longleftrightarrow} \forall x(\text{ck}(x) \to_{\mathbf{2}} \varphi(x))$$

$$\exists^W x\varphi(x) \stackrel{\text{def}}{\Longleftrightarrow} \exists x(\text{ck}(x) \wedge \varphi(x)).$$

For a formula φ of ZFC, φ^W denotes the formula obtained from φ by replacing relations $=$ and $\in$ by $=_{\mathbf{2}}$ and $\in_{\mathbf{2}}$, respectively, and quantifiers $\forall x$ and $\exists x$ by $\forall^W x$ and $\exists^W x$, respectively. By Theorem 4.4 and Corollary 4.6, φ^W is a global formula of LZFZ, hence a global formula of QtZFZ.

Since global formulas are subject to the laws of classical logic and it was proved in Titani [12] that LZFZ $\vdash A^W$ for each nonlogical axiom A of ZFC, we have:

Theorem 4.10 (Interpretation of ZFC in QtZFZ)**.** *If φ is a theorem of* ZFC, *then φ^W is provable in* QtZFZ.

4.5 Power set $\mathcal{P}(1)$ of 1

The power set $\mathcal{P}(1)$ of 1 ($= \{\emptyset\}$) is a global set and a complete lattice with respect to inclusion $\subset_{\mathbf{2}}$, where $a \subset_{\mathbf{2}} b \stackrel{\text{def}}{\Longleftrightarrow} \forall x(x \in_{\mathbf{2}} a \to_{\mathbf{2}} x \in_{\mathbf{2}} b)$.

$$a \leq b \stackrel{\text{def}}{\Longleftrightarrow} a \subset_{\mathbf{2}} b \quad \text{for } a, b \in_{\mathbf{2}} \mathcal{P}(1).$$

Then, for $\{x_\alpha\}_{\alpha \in_{\mathbf{2}} K} \subset_{\mathbf{2}} \mathcal{P}(1)$, where index set K is a check set,

$$\bigvee_{\alpha \in K} x_\alpha = \bigcup_{\alpha \in K} x_\alpha \in \mathcal{P}(1)$$
$$\bigwedge_{\alpha \in K} x_\alpha = \bigcap_{\alpha \in K} x_\alpha \in \mathcal{P}(1).$$

Hence, $\mathcal{P}(1)$ is a complete lattice. For $a, b \in \mathcal{P}(1)$, let

$$a\&b \stackrel{\text{def}}{=} \{x \in 1 \mid (0 \in a)\&(0 \in b)\}; \quad a \to_{\mathbf{2}} b \stackrel{\text{def}}{=} \{x \in 1 \mid (0 \in a) \to_{\mathbf{2}} (0 \in b)\};$$

$$a^* \stackrel{\text{def}}{=} \{x \in 1 \mid (0 \in a)^*\}; \quad a^\perp \stackrel{\text{def}}{=} \{x \in 1 \mid (0 \in a)^\perp\}; \quad \tilde{\mathfrak{e}} \stackrel{\text{def}}{=} \{x \in 1 \mid \mathfrak{e}\}.$$

Lemma 4.11. *The followings are provable in* QtZFZ, *where* $a, b \in \mathcal{P}(1)$ *and* $\{x_\alpha\}_{\alpha \in_{\mathbf{2}} K} \subset_{\mathbf{2}} \mathcal{P}(1)$.

1. $a\& \bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha =_{\mathbf{2}} \bigvee_{\alpha \in_{\mathbf{2}} K} a\&x_\alpha$,
2. $(\bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha)\&a =_{\mathbf{2}} \bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha\&a$,
3. $\tilde{\mathfrak{e}}\&a =_{\mathbf{2}} a\&\tilde{\mathfrak{e}} =_{\mathbf{2}} a$,
4. $a^{**} =_{\mathbf{2}} a$; $(a\&b)^* =_{\mathbf{2}} (b^*\&a^*)$; $(\bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha)^* =_{\mathbf{2}} (\bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha^*)$,
5. $a\&1 =_{\mathbf{2}} a \to_{\mathbf{2}} a\&a^*\&a =_{\mathbf{2}} a$.
6. $a\&1 =_{\mathbf{2}} a \to_{\mathbf{2}} a^{\perp\perp} =_{\mathbf{2}} a$,
7. $(a\&1 =_{\mathbf{2}} a) \wedge (b\&1 =_{\mathbf{2}} b) \wedge (a \leq_{\mathbf{2}} b) \to_{\mathbf{2}} (b =_{\mathbf{2}} (a \wedge b) \vee (a \wedge b^\perp))$.

Proof. 1.

$$\begin{aligned}
\vdash 0 \in_{\mathbf{2}} (a\& \bigvee_{\alpha \in_{\mathbf{2}} K} x_\alpha) \quad &\Leftrightarrow \quad (0 \in_{\mathbf{2}} a) \,\&\, \exists \alpha \in_{\mathbf{2}} K (0 \in_{\mathbf{2}} x_\alpha) \\
&\Leftrightarrow \quad \exists \alpha \in_{\mathbf{2}} K((0 \in_{\mathbf{2}} a) \,\&\, (0 \in_{\mathbf{2}} x_\alpha)) \\
&\Leftrightarrow \quad \exists \alpha \in_{\mathbf{2}} K(0 \in_{\mathbf{2}} a\&x_\alpha) \\
&\Leftrightarrow \quad 0 \in_{\mathbf{2}} \bigvee_{\alpha \in_{\mathbf{2}} K} (a\&x_\alpha)
\end{aligned}$$

2.–7. are similar. □

It follows that:

Theorem 4.12. QtZFZ $\vdash$ "$(\mathcal{P}(1), \subset, \bigcap, \bigcup, \&, ^*, \tilde{\mathfrak{e}})$ *is a Hilbert quantale" (cf. Definition 3.15).*

Since $\mathcal{P}(1)$ is a global set, there exists a check set Q and a bijection ρ such that $\rho : \mathcal{P}(1) \to Q$ by Theorems 4.9.

Definition 4.13. We fix a check set Q and a bijection ρ such that

$$\rho : \mathcal{P}(1) \to Q,$$

and define $\leq$, $\bigwedge$, $\bigvee$, $\&$, *, $\to_{\mathbf{2}}$ on Q so that

$$(Q, \leq, \bigwedge, \bigvee, \&, ^*, \mathfrak{e}) \cong (\mathcal{P}(1), \subset, \bigcap, \bigcup, \&, ^*, \tilde{\mathfrak{e}})$$

i.e.

$$\begin{aligned}
p \leq q &\overset{\text{def}}{\Longleftrightarrow} \rho^{-1}(p) \subset \rho^{-1}(q)\\
\bigwedge_\alpha p_\alpha &\overset{\text{def}}{=} \rho(\bigcap_\alpha \rho^{-1}(p_\alpha))\\
\bigvee_\alpha p_\alpha &\overset{\text{def}}{=} \rho(\bigcup_\alpha \rho^{-1}(p_\alpha))\\
p\&q &\overset{\text{def}}{\Longleftrightarrow} \rho(\rho^{-1}(p)\&\rho^{-1}(q))\\
p^* &\overset{\text{def}}{\Longleftrightarrow} \rho((\rho^{-1}(p))^*)\\
p \rightarrow_{\mathbf{2}} q &\overset{\text{def}}{=} \begin{cases} 1 & p \leq q \\ 0 & \text{otherwise} \end{cases}\\
\mathfrak{e} &\overset{\text{def}}{=} \rho(\tilde{\mathfrak{e}})
\end{aligned}$$

Definition 4.14. For a formula φ of QtZFZ, **inner truth value** $|\varphi|$ is defined by :

$$|\varphi| \overset{\text{def}}{=} \rho(\{t \in_{\mathbf{2}} 1 \mid \varphi\}).$$

Obviously we have:

Theorem 4.15.

1. $(Q, \leq, \bigwedge, \bigvee, \&, {}^*, \mathfrak{e})$ *is a Hilbert quantale, in* W*.*
2. $\mathrm{QtZFZ} \vdash (0 \in_{\mathbf{2}} \rho^{-1}|\varphi|) \Leftrightarrow \varphi$.

4.6 Model W^Q of QtZFZ constructed in QtZFZ

In the universe W constructed in QtZFZ, Hilbert quantale Q represents a truth value set of QtZFZ, and the relation $\prec$ defined by $\alpha \prec \beta \overset{\text{def}}{\Longleftrightarrow} (\alpha, \beta \in_{\mathbf{2}} \mathrm{On} \wedge \alpha \in_{\mathbf{2}} \beta)$ is a well founded relation with $\mathrm{Fld}(\prec) = \mathrm{On}$. Thus, recursive definition on On is justified in QtZFZ. Q-valued universe W^Q is constructed in W by $\in_{\mathbf{2}}$-recursion.

$$\begin{aligned}
W^Q_\alpha &= \{u \in_{\mathbf{2}} W \mid \left(\exists \beta \in_{\mathbf{2}} \alpha \exists \mathcal{D}u \subset_{\mathbf{2}} W^Q_\beta (\mathrm{Gl}(\mathcal{D}u) \wedge u : \mathcal{D}u \rightarrow Q)\right)^W\}\\
W^Q &= \bigcup_{\alpha \in_{\mathbf{2}} \mathrm{On}} W^Q_\alpha
\end{aligned}$$

Truth value of φ in W^Q will be denoted by $[\![\varphi]\!]^W_Q$. Atomic relations $=_{\mathbf{2}}$ and $\in_{\mathbf{2}}$ are interpreted in W^Q, as

$$\begin{aligned}
[\![x =_{\mathbf{2}} y]\!]^W_Q &= \bigwedge_{t \in_{\mathbf{2}} \mathcal{D}x} (x(t) \rightarrow_{\mathbf{2}} [\![t \in_{\mathbf{2}} y]\!]^W_Q) \wedge \bigwedge_{t \in_{\mathbf{2}} \mathcal{D}y} (y(t) \rightarrow_{\mathbf{2}} [\![t \in_{\mathbf{2}} x]\!]^W_Q)\\
[\![x \in_{\mathbf{2}} y]\!]^W_Q &= \bigvee_{t \in_{\mathbf{2}} \mathcal{D}y} [\![x =_{\mathbf{2}} t]\!]^W_Q \wedge y(t).
\end{aligned}$$

Logical operations $\wedge$, $\vee$, &, *, $^\perp$, $\to_{\mathbf{2}}$, $\forall$, $\exists$ are interpreted as the corresponding operations on Q. $\mathfrak{e}$ is interpreted as $\rho(\tilde{\mathfrak{e}})$. Then every sentence in W^Q has its truth value in Q.

Definition 4.16. For $x \in W^Q$, define $F(x)$ by

$$F(x) = \{F(t) \mid t \in_{\mathbf{2}} \mathcal{D}x \ \wedge \ 0 \in_{\mathbf{2}} \rho^{-1}(x(t))\}.$$

The following Lemma says that W^Q represents the universe of QtZFZ.

Lemma 4.17. *1.* QtZFZ $\vdash$ $\forall u \exists x \in W^Q (F(x) =_{\mathbf{2}} u)$.

2. QtZFZ $\vdash$ $\left(F(u) =_{\mathbf{2}} F(v) \leftrightarrow_{\mathbf{2}} 0 \in_{\mathbf{2}} [\![u =_{\mathbf{2}} v]\!]_Q^W\right) \wedge \left(F(u) \in_{\mathbf{2}} F(v) \leftrightarrow_{\mathbf{2}} 0 \in_{\mathbf{2}} [\![u \in_{\mathbf{2}} v]\!]_Q^W\right)$.

Proof. The proof is carried out in QtZFZ.

1. By $\in$-induction. Let $\Psi(u, \alpha) \overset{\text{def}}{\Longleftrightarrow} \exists x \in W_\alpha^Q (u =_{\mathbf{2}} F(x))$. Then by using A8 (Collection) we have

$$\exists \gamma \in_{\mathbf{2}} \mathrm{On} \left(\forall v \in_{\mathbf{2}} u \exists \beta \Psi(v, \beta)) \to_{\mathbf{2}} \forall v \in_{\mathbf{2}} u \exists \beta \in_{\mathbf{2}}^{\square} \gamma \exists y \in_{\mathbf{2}} W_\beta^Q (v =_{\mathbf{2}} F(y))\right).$$

   Let

$$\begin{cases} \mathcal{D}x \overset{\text{def}}{=} W_\gamma^Q \\ x(y) \overset{\text{def}}{=} \rho(\{t \in 1 \mid F(y) \in u\}) \end{cases}$$

   Then $x \in_{\mathbf{2}} W_{\gamma+1}^L$ and $F(x) =_{\mathbf{2}} u$.

$$\begin{array}{rrcl} \because) & z \in_{\mathbf{2}} F(x) & \Rightarrow & \exists t \in_{\mathbf{2}} \mathcal{D}x (z =_{\mathbf{2}} F(t) \wedge 0 \in_{\mathbf{2}} \rho^{-1}(x(t))) \\ & & & 0 \in_{\mathbf{2}} \rho^{-1}(x(t)) \Leftrightarrow F(t) \in_{\mathbf{2}} u \\ & & \Rightarrow & z \in_{\mathbf{2}} u. \\ & z \in_{\mathbf{2}} u & \Rightarrow & \exists \beta \in_{\mathbf{2}}^{\square} \gamma \Psi(z, \beta) \\ & & \Rightarrow & \exists s \in_{\mathbf{2}} W_\beta^Q (z =_{\mathbf{2}} F(s) \in_{\mathbf{2}} u) \\ & & \Rightarrow & \exists s \in_{\mathbf{2}} W_\beta^Q (z =_{\mathbf{2}} F(s) \ \wedge \ 0 \in_{\mathbf{2}} \rho^{-1}(x(s))) \\ & & \Rightarrow & z \in_{\mathbf{2}} F(x). \end{array}$$

   It follows that

$$\forall v \in_{\mathbf{2}} u \exists \beta \Psi(v, \beta)) \Rightarrow \exists \alpha \Psi(u, \alpha)$$

   That is, $\forall u \exists x (F(x) =_{\mathbf{2}} u)$.

2. By induction.

$$\begin{array}{rcl} \left(0 \in_{\mathbf{2}} \rho^{-1}([\![u =_{\mathbf{2}} v]\!]_Q^W\right), (F(t) \in_{\mathbf{2}} F(u)) & \Rightarrow & (t \in_{\mathbf{2}} \mathcal{D}u) \wedge \left(0 \in_{\mathbf{2}} \rho^{-1}(u(t))\right) \\ & \Rightarrow & 0 \in_{\mathbf{2}} \rho^{-1} [\![t \in v]\!]_Q^W \\ & \Rightarrow & F(t) \in_{\mathbf{2}} F(v) \end{array}$$

Since $0 \in_2 \rho^{-1}([\![u =_2 v]\!]^W_Q)$ is global, we have

$$0 \in_2 \rho^{-1}[\![u =_2 v]\!]^W_Q \quad \Rightarrow \quad F(u) =_2 F(v)$$

$$\begin{aligned} \big(F(u) =_2 F(v) \,\wedge\, t \in_2 \mathcal{D}u \wedge 0 \in_2 \rho^{-1}(u(t))\big) &\Rightarrow F(t) \in_2 F(v) \\ &\Rightarrow \exists s \in_2 \mathcal{D}y \,\big(F(t) =_2 F(s) \,\wedge\, 0 \in_2 \rho^{-1} v(s))\big) . \end{aligned}$$

$$\therefore \quad F(u) =_2 F(v) \Rightarrow 0 \in_2 \rho^{-1}[\![u =_2 v]\!]^W_Q .$$

$$\begin{aligned} 0 \in_2 \rho^{-1}[\![u \in_2 v]\!]^W_Q \quad &\Longleftrightarrow \quad \exists t \in_2 \mathcal{D}v \,\big(\, 0 \in_2 \rho^{-1}([\![u =_2 t]\!]^W_Q \wedge v(t))\,\big) \\ &\Longleftrightarrow \quad F(u) =_2 F(t) \in_2 F(v) \end{aligned}$$

□

The following theorem shows that QtZFZ is represented by the set theory on W^Q.

Theorem 4.18. *For every sentence φ of* QtZFZ*,*

$$\text{QtZFZ} \vdash [\![\varphi]\!]^W_Q =_2 |\varphi|,$$

Proof. It suffices to show that

$$\text{QtZFZ} \vdash \; [\![\varphi(u_1, \cdots, u_n)]\!]^W_Q =_2 |\varphi(F(u_1), \cdots, F(u_n))|,$$

for every formula $\varphi(a_1, \cdots, a_n)$ of QtZFZ and for $u_1, \cdots u_n \in_2 W^Q$, by induction on the complexity of φ.

1. By Lemma 4.17 and Theorem 4.15,

$$\text{QtZFZ} \vdash \; \big(|F(u) =_2 F(v)| =_2 [\![u =_2 v]\!]^W_Q\big) \,\wedge\, \big(|F(u) \in_2 F(v)| =_2 [\![u \in_2 v]\!]^W_Q\big) .$$

2. If φ is of the form $\varphi_1 \wedge \varphi_2$, then

$$\begin{aligned} \text{QtZFZ} \vdash 0 \in_2 \rho^{-1}|\varphi \wedge \psi| \quad &\Leftrightarrow \quad \varphi \wedge \psi \\ &\Leftrightarrow \quad 0 \in_2 \rho^{-1}|\varphi| \wedge 0 \in_2 \rho^{-1}|\psi| \\ &\Leftrightarrow \quad 0 \in_2 \rho^{-1}|\varphi| \cap \rho^{-1}|\psi| \\ &\Leftrightarrow \quad 0 \in_2 \rho^{-1}(|\varphi| \wedge |\psi|) \end{aligned}$$

3. If φ is of the form $\varphi_1 \vee \varphi_2$, then similarly to (2).

4. If φ is of the form $\varphi_1 \& \varphi_2$, then

$$\begin{aligned} \text{QtZFZ} \vdash 0 \in_{\mathbf{2}} \rho^{-1}|\varphi \& \psi| &\Leftrightarrow \varphi \,\&\, \psi \\ &\Leftrightarrow (0 \in_{\mathbf{2}} \rho^{-1}|\varphi|) \,\&\, (0 \in_{\mathbf{2}} \rho^{-1}|\psi|) \\ &\Leftrightarrow 0 \in_{\mathbf{2}} (\rho^{-1}|\varphi| \,\&\, \rho^{-1}|\psi|) \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}(|\varphi| \,\&\, |\psi|) \end{aligned}$$

5. If φ is of the form $\varphi_1^{\perp}$,

$$\begin{aligned} \text{QZFZ} \vdash 0 \in_{\mathbf{2}} \rho^{-1}|\varphi_1^{\perp}| &\Leftrightarrow \varphi_1^{\perp} \\ &\Leftrightarrow (0 \in_{\mathbf{2}} \rho^{-1}|\varphi_1|)^{\perp} \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}(|\varphi_1|^{\perp}) \end{aligned}$$

6. If φ is of the form $\varphi_1 \to_{\mathbf{2}} \varphi_2$, then similarly

$$\begin{aligned} \text{QtZFZ} \vdash 0 \in \rho^{-1}|\varphi_1 \to_{\mathbf{2}} \varphi_2| &\Leftrightarrow \varphi_1 \to_{\mathbf{2}} \varphi_2 \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}|\varphi_1| \to_{\mathbf{2}} 0 \in_{\mathbf{2}} \rho^{-1}|\varphi_2| \\ &\Leftrightarrow \rho^{-1}|\varphi_1| \subset_{\mathbf{2}} \rho^{-1}|\varphi_2| \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}(|\varphi_1| \to_{\mathbf{2}} |\varphi_2|) \end{aligned}$$

7. If $\varphi(x_1, \cdots, x_n)$ is of the form $\forall x \psi(x, x_1, \cdots, x_n)$, then by Lemma 4.17,

$$\begin{aligned} \text{QtZFZ} \vdash 0 \in_{\mathbf{2}} \rho^{-1}[\![\varphi]\!]_Q^W &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}(\bigwedge_x [\![\psi(x, x_1, \cdots, x_n)]\!]_Q^W) \\ &\Leftrightarrow \forall x(\psi(F(x), F(x_1), \cdots, F(x_n)) \\ &\Leftrightarrow \forall z \psi(z, F(x_1), \cdots, F(x_n)). \end{aligned}$$

8. If φ is of the form $\exists x \psi(x, x_1, \cdots, x_n)$, then similar to (6).
9. If φ is of the form ψ^*,

$$\begin{aligned} \text{QZFZ} \vdash 0 \in_{\mathbf{2}} \rho^{-1}|\psi^*| &\Leftrightarrow \psi^* \\ &\Leftrightarrow (0 \in_{\mathbf{2}} \rho^{-1}|\psi|)^* \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}(|\psi|^*). \end{aligned}$$

10. If φ is $\mathfrak{e}$,

$$\begin{aligned} \text{QZFZ} \vdash 0 \in_{\mathbf{2}} \rho^{-1}|\mathfrak{e}| &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}\rho(\{x \in 1 \mid \mathfrak{e}\}) \Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}\rho(\tilde{\mathfrak{e}}) \\ &\Leftrightarrow 0 \in_{\mathbf{2}} \rho^{-1}[\![\mathfrak{e}]\!]_Q^W. \end{aligned}$$

□

Corollary 4.19. QtZFZ $\vdash \Box_{\mathbf{2}}\varphi \Leftrightarrow$ "$[\![\varphi]\!]_Q^W =_{\mathbf{2}} 1$ *in* W^Q ".

4.7 Proof of a completeness of QtZFZ

For a sentence φ of quantale valued set theory,

$$\begin{aligned} \text{ZFC} \quad &\vdash \quad \text{“ } [\![\varphi]\!]_Q = 1 \text{ in } V^Q \text{ for all Hilbert quantale } Q \text{ ”} \\ &\Longrightarrow \quad \text{QtZFZ} \vdash \varphi, \end{aligned}$$

where V^Q is the Q-valued universe constructed in ZFC.

Proof. Suppose that φ is a sentence of QtZFZ and

$$\text{ZFC} \vdash \text{ “ } [\![\varphi]\!]_Q = 1 \text{ in } V^Q \text{ for all Hilbert quantale } Q \text{ ”}.$$

Then, since W is a model of ZFC and Q is a Hilbert quantale in W,

$$\text{QtZFZ} \vdash \text{ “ } [\![\varphi]\!]_Q^W =_{\mathbf{2}} 1 \ \text{ in } W^Q \text{ ”}.$$

Hence, by Corollary 4.19,

$$\text{QtZFZ} \vdash [\![\varphi]\!]_Q^W =_{\mathbf{2}} |\varphi| =_{\mathbf{2}} 1.$$

$$\therefore \quad \text{QtZFZ} \vdash \varphi.$$

□

5 Real and complex numbers

Let V be a universe of ZFC and $\mathcal{L}$ be an arbitrary complete lattice in V. It is known that each rational number defined in $V^{\mathcal{L}}$ is a check set, and the check set $\check{\mathbb{Q}}$ associated with the set $\mathbb{Q}$ of rational numbers in V is the set of rational numbers in $V^{\mathcal{L}}$.

$$\mathcal{D}\check{\mathbb{Q}} = \{\check{r} \mid r \in \mathbb{Q}\}, \quad \check{\mathbb{Q}}(\check{r}) = 1 \ \text{ for } r \in \mathbb{Q}.$$

5.1 Definition of real numbers

If $\mathcal{L}$ is an arbitrary complete orthomodular lattice, then real numbers in $V^{\mathcal{L}}$ are defined as Dedekind cuts of $\check{\mathbb{Q}}$.

Definition 5.1. 'u is a **real number**' if $(D\,0)_u \wedge (D1)_u \wedge (D2)_u$:

$$\begin{aligned} &(D\,0)_u \quad u \subset_{\mathbf{2}} \check{\mathbb{Q}}, \quad \text{ i.e. } \forall x (x \in_{\mathbf{2}} u \to_{\mathbf{2}} x \in_{\mathbf{2}} \check{\mathbb{Q}}), \\ &(D\,1)_u \quad \exists x \in_{\mathbf{2}} \check{\mathbb{Q}} (x \in_{\mathbf{2}} u) \wedge \exists x \in_{\mathbf{2}} \check{\mathbb{Q}} (x \in_{\mathbf{2}} u)^{\perp}, \\ &(D\,2)_u \quad \forall x \in_{\mathbf{2}} \check{\mathbb{Q}} \left(x \in_{\mathbf{2}} u \leftrightarrow_{\mathbf{2}} \forall y \in_{\mathbf{2}} \check{\mathbb{Q}} (x < y \to_{\mathbf{T}} y \in_{\mathbf{2}} u) \right), \\ &\qquad\qquad \text{where } \varphi \to_{\mathbf{T}} \psi \overset{\text{def}}{\Longleftrightarrow} \varphi^{\perp} \vee (\varphi \wedge \psi). \end{aligned}$$

The set of real numbers is denoted by $\mathfrak{R}$.

$$\mathfrak{R} \overset{\text{def}}{=} \{u \mid (D\,0)_u \wedge (D\,1)_u \wedge (D\,2)_u\}.$$

Lemma 5.2. *Let $\mathcal{L}$ be a complete orthomodular lattice. If $[\![u \subset_2 \check{\mathbb{Q}}]\!]_{\mathcal{L}} \neq 0$, then there exists $v \in V^{\mathcal{L}}$ such that*

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\} \quad \text{and} \quad [\![u =_2 v]\!]_{\mathcal{L}} = 1.$$

Proof. If $[\![u \subset_2 \check{\mathbb{Q}}]\!]_{\mathcal{L}} \neq 0$, then $[\![u \subset_2 \check{\mathbb{Q}}]\!]_{\mathcal{L}} = 1$, since $u \subset_2 \check{\mathbb{Q}}$ is a global formula. Define v by

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\}, \quad v(\check{r}) = [\![\check{r} \in_2 u]\!]_{\mathcal{L}} \quad \text{for } r \in \mathbb{Q}.$$

Then for $x \in \mathcal{D}u$,

$$\begin{aligned} u(x) &= u(x) \wedge [\![u \subset_2 \check{\mathbb{Q}}]\!]_{\mathcal{L}} \leq u(x) \wedge [\![x \in_2 \check{\mathbb{Q}}]\!]_{\mathcal{L}} = \bigvee_{r \in \mathbb{Q}} (u(x) \wedge [\![x =_2 \check{r}]\!]_{\mathcal{L}}) \\ &= \bigvee_{r \in \mathbb{Q}} [\![x =_2 \check{r} \wedge \check{r} \in_2 u]\!]_{\mathcal{L}} = \bigvee_{r \in \mathbb{Q}} ([\![x =_2 \check{r}]\!]_{\mathcal{L}} \wedge v(\check{r})) \leq [\![x \in_2 v]\!]_{\mathcal{L}}. \end{aligned}$$

Therefore, $[\![u =_2 v]\!]_{\mathcal{L}} = 1$. □

Power set $\mathcal{P}(\check{\mathbb{Q}})$ of $\check{\mathbb{Q}}$ in $V^{\mathcal{L}}$ is defined by

$$\mathcal{D}\mathcal{P}(\check{\mathbb{Q}}) = \{u \in V^{\mathcal{L}} \mid \mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}\} \quad \text{and} \quad \mathcal{P}(\check{\mathbb{Q}})(u) = 1 \ \text{ for } u \in \mathcal{D}\mathcal{P}(\check{\mathbb{Q}}).$$

Then $[\![\forall u(u \in_2 \mathcal{P}(\check{\mathbb{Q}}) \leftrightarrow_2 u \subset_2 \check{\mathbb{Q}})]\!]_{\mathcal{L}} = 1$ (cf. [12]).

The set of real numbers with domain $\{\check{r} \mid r \in \mathbb{Q}\}$ in $V^{\mathcal{L}}$ is denoted by $\mathfrak{R}_{\mathcal{L}}$.

$$\mathfrak{R}_{\mathcal{L}} \stackrel{\text{def}}{=} \{u \in V^{\mathcal{L}} \mid \mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\} \text{ and } [\![u \text{ is a real number }]\!]_{\mathcal{L}} = 1\}.$$

Lemma 5.3. *Let $\mathcal{L}$ be a complete orthomodular lattice.*

1. *If $p, q \in \mathcal{L}$, then $(p \rightarrow_{\mathbf{T}} q) = 1 \iff (p \rightarrow_2 q) = 1 \iff p \leq q$.*
2. *If $u \in \mathfrak{R}_{\mathcal{L}}$ and $r \in \mathbb{Q}$, then $[\![\check{r} \in_2 u]\!]_{\mathcal{L}} = [\![\check{r} \in_{\mathbf{T}} u]\!]_{\mathcal{L}} = u(\check{r})$.*
3. *If $u, v \in \mathfrak{R}_{\mathcal{L}}$, then $[\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}} = 1 \iff [\![u =_2 v]\!]_{\mathcal{L}} = 1$.*
4. *Let B be a Boolean $(\bigvee, ^{\perp})$-sublattice of $\mathcal{L}$. If $p \in B$, $u \in V^B$ and $p \leq [\![u \subset_{\mathbf{T}} \check{\mathbb{Q}}]\!]_{\mathcal{L}}$, then there exists $v \in V^B$ such that*

   $$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\} \quad \text{and} \quad p \leq [\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}},$$

   where $u \subset_{\mathbf{T}} v \stackrel{def}{\iff} \forall x(x \in_{\mathbf{T}} u \rightarrow_{\mathbf{T}} x \in_{\mathbf{T}} v)$.

Proof. 1. By Theorem 2.9, $p \wedge (p \rightarrow_{\mathbf{T}} q) = p \wedge (p^{\perp} \vee (p \wedge q)) = p \wedge q$.
Hence, $(p \rightarrow_{\mathbf{T}} q) = 1 \iff p \leq q$.

2. $[\![\check{r} =_{\mathbf{T}} \check{s}]\!]_{\mathcal{L}} = 1 \Longleftrightarrow [\![\check{r} =_{\mathbf{2}} \check{s}]\!]_{\mathcal{L}} = 1 \Longleftrightarrow r = s,$ for $r, s \in \mathbb{Q}$.
Hence, if $\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}$, then for $r \in \mathbb{Q}$,

$$[\![\check{r} \in_{\mathbf{T}} u]\!]_{\mathcal{L}} = \bigvee_{s \in \mathbb{Q}} [\![\check{r} =_{\mathbf{T}} \check{s}]\!]_{\mathcal{L}} \wedge u(\check{s}) = \bigvee_{s \in \mathbb{Q}} [\![\check{r} =_{\mathbf{2}} \check{s}]\!]_{\mathcal{L}} \wedge u(\check{s}) = u(\check{r}) = [\![\check{r} \in_{\mathbf{2}} u]\!]_{\mathcal{L}}.$$

3. If $\mathcal{D}u = \mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\}$, then, by 1. and 2.,

$$\begin{aligned} [\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}} \quad &= \quad \bigwedge_{r \in \mathbb{Q}} (u(\check{r}) \rightarrow_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} v]\!]_{\mathcal{L}}) \wedge \bigwedge_{r \in \mathbb{Q}} (v(\check{r}) \rightarrow_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]_{\mathcal{L}}) = 1 \\ &\Longleftrightarrow \quad [\![u =_{\mathbf{2}} v]\!]_{\mathcal{L}} = 1. \end{aligned}$$

4. For $u \in V^B$ such that $[\![u \subset_{\mathbf{T}} \check{\mathbb{Q}}]\!]_{\mathcal{L}} = p$, define v by

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\}, \quad v(\check{r}) = [\![\check{r} \in_{\mathbf{T}} u]\!]_{\mathcal{L}}.$$

If $x \in \mathcal{D}u$, then $p \wedge u(x) \leq u(x) \wedge [\![u \subset_{\mathbf{T}} \check{\mathbb{Q}}]\!]_{\mathcal{L}} \leq [\![x \in_{\mathbf{T}} \check{\mathbb{Q}}]\!]_{\mathcal{L}} = \bigvee_{r \in \mathbb{Q}} [\![x =_{\mathbf{T}} \check{r}]\!]_{\mathcal{L}}$.
By logical distributive law and equality axioms for $=_{\mathbf{T}}$ in the Boolean valued universe V^B,

$$\begin{aligned} p \wedge u(x) \quad &\leq \quad \bigvee_{r \in \mathbb{Q}} u(x) \wedge [\![x =_{\mathbf{T}} \check{r}]\!]_{\mathcal{L}} \leq \bigvee_{r \in \mathbb{Q}} [\![x =_{\mathbf{T}} \check{r} \wedge \check{r} \in_{\mathbf{T}} u]\!]_{\mathcal{L}} = [\![x \in_{\mathbf{T}} v]\!]_{\mathcal{L}}. \\ \therefore \qquad & p \leq \bigwedge_{x \in \mathcal{D}u} (u(x) \rightarrow_{\mathbf{T}} [\![x \in_{\mathbf{T}} v]\!]_{\mathcal{L}}). \\ & \bigwedge_{x \in \mathcal{D}v} (v(x) \rightarrow_{\mathbf{T}} [\![x \in_{\mathbf{T}} u]\!]_{\mathcal{L}}) = 1 \text{ is obvious.} \\ \therefore \qquad & p \leq [\![u =_{\mathbf{T}} v]\!]_{\mathcal{L}}. \end{aligned}$$

□

5.2 Real numbers in $V^{P(\mathcal{H})}$

If B is a Boolean $(\bigvee, ^{\perp})$-sublattice of $P(\mathcal{H})$ (cf. Definition 2.2), then the identity mapping $B \to P(\mathcal{H})$ is an injection preserving $\bigvee$ and $^{\perp}$. Hence, if $\varphi(x_1, x_2, \cdots, x_n)$ is a bounded formula of quantum set theory (cf. Definition 2.28), and if $u_1, u_2, \cdots, u_n \in V^B$, then

$$[\![\varphi(u_1, u_2, \cdots, u_n)]\!]_{P(\mathcal{H})} = [\![\varphi(u_1, u_2, \cdots, u_n)]\!]_B$$

by Theorem 2.36.

Theorem 5.4. *If u is a real number in $V^{P(\mathcal{H})}$, i.e. $u \in \mathfrak{R}_{P(\mathcal{H})}$, then there exists a Boolean $(\bigvee, ^{\perp})$-sublattice B of $P(\mathcal{H})$ such that $u \in V^B$.*

Proof. If $u \in \mathfrak{R}_{P(\mathcal{H})}$, then $\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}$. By $(D2)_u$ of the Definition 5.1,

$$r \leq s \Longrightarrow [\![\check{r} \in_{\mathbf{2}} u]\!]_{P(\mathcal{H})} \leq [\![\check{s} \in_{\mathbf{2}} u]\!]_{P(\mathcal{H})}.$$

Hence, $M = \{[\![\check{r} \in_{\mathbf{2}} u]\!] \mid r \in \mathbb{Q}\}$ is compatible. Let $B = M''$, where

$$M' = \{p \in P(\mathcal{H}) \mid p \,\lvert_{\circ}\, q \quad \forall q \in M\}.$$

B is a Boolean $(\bigvee, ^{\perp})$-sublattice of $P(\mathcal{H})$ by Theorem 2.10, and $u \in V^B$. □

5.2.1 Takeuti's definition of real numbers

In [10], G.Takeuti defined real numbers in a Boolean valued sub-universe V^B of $V^{P(\mathcal{H})}$, as follows:

$$u \text{ is a } \textbf{real number} \overset{\text{def}}{\Longleftrightarrow} (TD\,0)_u \wedge (TD\,1)_u \wedge (TD\,2)_u :$$

$$\begin{array}{ll} (TD\,0)_u & u \subset \check{\mathbb{Q}}, \\ (TD\,1)_u & \exists x \in \check{\mathbb{Q}}(x \in u) \wedge \exists x \in \check{\mathbb{Q}}((x \in u)^{\perp}) \\ (TD\,2)_u & \forall x \in \check{\mathbb{Q}}\left(x \in u \leftrightarrow \forall t \in \check{\mathbb{Q}}(x < t \to t \in u)\right). \end{array}$$

Theorem 5.5. *Takeuti's definition of real numbers is equivalent to our definition in the following sense.*

If $u \in V^B$ for some Boolean $(\bigvee, {}^{\perp})$-sublattice B of $P(\mathcal{H})$, then

$$[\![(TD\,0)_u \wedge (TD\,1)_u \wedge (TD\,2)_u]\!]^T \quad = \quad [\![u \in_{\mathbf{T}} \mathfrak{R}]\!]_{P(\mathcal{H})},$$

where $[\![u \in_{\mathbf{T}} \mathfrak{R}]\!]_{P(\mathcal{H})} = [\![\exists v[(u =_{\mathbf{T}} v) \wedge (D\,0)_v \wedge (D\,1)_v \wedge (D\,2)_v]]\!]_{P(\mathcal{H})}$.

Proof. By Theorem 2.41,

$$[\![(TD\,0)_u \wedge (TD\,1)_u \wedge (TD\,2)_u]\!]^T = [\![(TD\,0)_u^{(T)} \wedge (TD\,1)_u^{(T)} \wedge (TD\,2)_u^{(T)}]\!]_{P(\mathcal{H})},$$

where $\varphi^{(T)}$ is the formula obtained from φ by replacing $=$, $\in$, $\to$, $\forall x \in u$, $\exists x \in u$ by $=_{\mathbf{T}}$, $\in_{\mathbf{T}}$, $\to_{\mathbf{T}}$, $\forall x \in_{\mathbf{T}} u$, $\exists x \in_{\mathbf{T}} u$ respectively. $\varphi^{(T)}$ is a formula of QZFZ. Since $u \in V^B$, it suffices to show in V^B, by Theorem 2.36, that

$$[\![(TD\,0)_u^{(T)} \wedge (TD\,1)_u^{(T)} \wedge (TD\,2)_u^{(T)}]\!]_B = [\![u \in_{\mathbf{T}} \mathfrak{R}]\!]_B.$$

Let $p = [\![(TD\,0)_u^{(T)} \wedge (TD\,1)_u^{(T)} \wedge (TD\,2)_u^{(T)}]\!]_B$, and define $v \in V^B$ by

$$\mathcal{D}v = \{\check{r} \mid r \in \mathbb{Q}\} \quad \text{and} \quad v(\check{r}) = ([\![\check{r} \in_{\mathbf{T}} u]\!] \wedge p) \vee ([\![\check{r} \in_{\mathbf{T}} \check{0}]\!] \wedge p^{\perp}).$$

Then, by Lemma 5.3, $[\![\check{r} \in_{\mathbf{T}} v]\!]_B = [\![\check{r} \in_{\mathbf{2}} v]\!]_B = v(\check{r})$, and

1. $p \leq [\![u =_{\mathbf{T}} v]\!]_B$.

$$\begin{array}{lll} \because) & p \wedge v(\check{r}) \leq [\![\check{r} \in_{\mathbf{T}} u]\!]_B & \text{by Definition.} \\ & p \wedge u(x) \leq p \wedge u(x) \wedge [\![u \subset_{\mathbf{T}} \check{\mathbb{Q}}]\!]_B \leq [\![x \in_{\mathbf{T}} \check{\mathbb{Q}}]\!]_B & \text{for } x \in \mathcal{D}u \\ \therefore & p \wedge u(x) \leq \bigvee_{r \in \mathbb{Q}} [\![x =_{\mathbf{T}} \check{r}]\!]_B \wedge u(x) \wedge p \leq [\![x \in_{\mathbf{T}} v]\!]_B. & \end{array}$$

2. $[\![(D\,0)_v]\!]_B = [\![\forall x(x \in_{\mathbf{2}} v \to_{\mathbf{2}} x \in_{\mathbf{2}} \check{\mathbb{Q}})]\!]_B = 1.$ Obvious.

3. $[\![(D\,1)_v]\!] = \bigvee_{r \in \mathbb{Q}} [\![\check{r} \in_{\mathbf{2}} v]\!]_B \wedge \bigvee_{r \in \mathbb{Q}} [\![\check{r} \in_{\mathbf{2}} v]\!]_B^{\perp} = 1.$

Proof.

$$p = p \wedge [\![(TD\,1)_u^{(T)}]\!] \leq \bigvee_{r\in\mathbb{Q}} p \wedge [\![\check{r} \in_{\mathbf{T}} u]\!]_B \leq \bigvee_{r\in\mathbb{Q}} v(\check{r}).$$

If $r \geq 0$, then $[\![\check{r} \in \check{0}]\!]_B = 1$, so $p^\perp \leq v(\check{r})$. Hence, $p^\perp \leq \bigvee_{r\in\mathbb{Q}} v(\check{r})$.

$$\therefore 1 = p \vee p^\perp \leq \bigvee_{r\in\mathbb{Q}} v(\check{r}).$$

If $r < 0$, then $[\![\check{r} \in \check{0}]\!]_B = 0$. Hence, $v(\check{r}) = [\![\check{r} \in_{\mathbf{T}} u]\!]_B \wedge p$ for $r < 0$.

$$\therefore\ v(\check{r})^\perp = [\![\check{r} \in_{\mathbf{T}} u]\!]_B^\perp \vee p^\perp \geq p^\perp \text{ for } r < 0,$$

$$\therefore\ p^\perp \leq \bigvee_{r\in\mathbb{Q}} v(\check{r})^\perp. \tag{5.1}$$

$p \leq \bigvee_{r\in\mathbb{Q}} [\![\check{r} \in_{\mathbf{T}} u]\!]_B^\perp \leq \bigvee_{r\in\mathbb{Q}} ([\![\check{r} \in_{\mathbf{T}} u]\!]_B \wedge p)^\perp$ and
$p \leq \bigwedge_{r\in\mathbb{Q}} (\check{0}(\check{r})^\perp \vee p) = \bigwedge_{r\in\mathbb{Q}} (\check{0}(\check{r}) \wedge p^\perp)^\perp$.

$$\begin{aligned}\therefore\quad p \quad &\leq \quad \bigvee_{r\in\mathbb{Q}} ([\![\check{r} \in_{\mathbf{T}} u]\!]_B \wedge p)^\perp \wedge \bigwedge_{r\in\mathbb{Q}} (\check{0}(\check{r}) \wedge p^\perp)^\perp \\ &\leq \quad \bigvee_{r\in\mathbb{Q}} \big(([\![\check{r} \in_{\mathbf{T}} u]\!]_B \wedge p) \vee (\check{0}(\check{r}) \wedge p^\perp)\big)^\perp = \bigvee_{r\in\mathbb{Q}} v(\check{r})^\perp\end{aligned}$$

$$\therefore\quad p \leq \bigvee_{r\in\mathbb{Q}} v(\check{r})^\perp \tag{5.2}$$

By (5.1) and (5.2), $1 = p \vee p^\perp \leq \bigvee_{r\in\mathbb{Q}} v(\check{r})^\perp$. □

4. $[\![(D\,2)_v]\!] = [\![\forall x \in_{\mathbf{2}} \check{\mathbb{Q}} \left(x \in_{\mathbf{2}} v \leftrightarrow_{\mathbf{2}} \forall y \in_{\mathbf{2}} \check{\mathbb{Q}}(x < y \to_{\mathbf{T}} y \in_{\mathbf{2}} v)\right)]\!]_B = 1$

$$\begin{aligned}\because)\quad p \leq [\![(TD\,2)_u^{(T)}]\!]_B \quad &\iff \quad p \leq \bigwedge_{r\in\mathbb{Q}} \left([\![\check{r} \in_{\mathbf{T}} u]\!]_B \leftrightarrow_T \bigwedge_{s\in\mathbb{Q},\, r<s} [\![\check{s} \in_{\mathbf{T}} u]\!]_B \right) \\ &\iff \quad \left(\bigwedge_{r\in\mathbb{Q}} [\![\check{r} \in_{\mathbf{2}} u]\!]_B \wedge p \leftrightarrow_{\mathbf{2}} \bigwedge_{s\in\mathbb{Q},\, r<s} [\![\check{s} \in_{\mathbf{2}} u]\!]_B \wedge p \right) = 1 \\ &\iff \quad \left(\bigwedge_{r\in\mathbb{Q}} v(\check{r}) \leftrightarrow_{\mathbf{2}} \bigwedge_{s\in\mathbb{Q},\, r<s} v(\check{s}) \right) = [\![(D\,2)_v]\!]_B = 1\end{aligned}$$

By 1.–4., we have $p \leq [\![(u =_{\mathbf{T}} v) \wedge (v \in_{\mathbf{2}} \mathfrak{R})]\!]_B = [\![u \in_{\mathbf{T}} \mathfrak{R}]\!]_B$.

Conversely, by Lemma 5.3,

$[\![(D0)_v \wedge (D1)_v \wedge (D2)_v]\!]_B \leq [\![(TD0)_v^{(T)} \wedge (TD1)_v^{(T)} \wedge (TD2)_v^{(T)}]\!]_B$, and by equality axiom in V^B,

$$[\![\exists v[(u =_{\mathbf{T}} v)\wedge(TD0)_v^{(T)}\wedge(TD1)_v^{(T)}\wedge(TD2)_v^{(T)}]]\!]_B \leq [\![(TD0)_u^{(T)}\wedge(TD1)_u^{(T)}\wedge(TD2)_u^{(T)}]\!]_B$$

$$\therefore \quad [\![u \in_{\mathbf{T}} \mathfrak{R}]\!]_B \quad = \quad [\![(TD0)_u^{(T)} \wedge (TD1)_u^{(T)} \wedge (TD2)_u^{(T)}]\!]_B.$$

□

Theorem 5.6 (Takeuti [10]). *If u is a real number in $V^{P(\mathcal{H})}$, then mapping $E_u : \mathbb{R} \to P(\mathcal{H})$ defined by*

$$E_u(\lambda) = \bigwedge_{\lambda < r} [\![\check{r} \in_{\mathbf{T}} u]\!]^T = \bigwedge_{\lambda < r} u(\check{r})$$

is a spectral measure, i.e.

$$\bigwedge_{\lambda\in\mathbb{R}} E_u(\lambda) = 0 \qquad \bigvee_{\lambda\in\mathbb{R}} E_u(\lambda) = 1 \qquad E_u(\lambda) = \bigwedge_{\lambda<\mu} E_u(\mu)\,,$$

which gives the spectral representation of self-adjoint operator on the Hilbert space $\mathcal{H}$:

$$A = \int \lambda dE_u(\lambda).$$

Conversely, if A is a self-adjoint operator on $\mathcal{H}$, $A = \int \lambda dE(\lambda)$, then $u \in V^{P(\mathcal{H})}$ defined by

$$\mathcal{D}u = \{\check{r} \mid r \in \mathbb{Q}\}, \qquad u(\check{r}) = \bigwedge_{r<s} E(s)$$

is a real number in $V^{P(\mathcal{H})}$, and $A = \int \lambda dE_u(\lambda)$.

The real number u in $V^{P(\mathcal{H})}$ such that $A = \int \lambda dE_u(\lambda)$ is denoted by $\widehat{A}$.

Definition 5.7. Real numbers u, v are said to be **compatible**, in symbols $u \,\lfloor_{\circ}\, v$, if $\forall r, s \in \check{\mathbb{Q}}(r \in_{\mathbf{2}} u \,\lfloor_{\circ}\, s \in_{\mathbf{2}} v)$ (cf. Definition 2.31). Then

$$[\![u \,\lfloor_{\circ}\, v]\!]_{P(\mathcal{H})} = 1 \iff \lfloor_{\circ}(\{u(\check{r}) \mid r \in \mathbb{Q}\} \cup \{v(\check{r}) \mid r \in \mathbb{Q}\}).$$

If $[\![u \,\lfloor_{\circ}\, v]\!]_{P(\mathcal{H})} = 1$, then by Theorem 2.10, there exists a Boolean $(\bigvee, {}^{\perp})$-sublattice B of $P(\mathcal{H})$ such that $u, v \in V^B$.

Theorem 5.8 (Takeuti [10]). *Let B be a Boolean $(\bigvee, {}^{\perp})$-sublattice of $P(\mathcal{H})$, and let u, v be real numbers in V^B. Then for $p \in B$,*

$$p \leq [\![u =_{\mathbf{T}} v]\!]_B \iff A_u \cdot p = A_v \cdot p,$$

where A_u and A_v are self-adjoint operators on $\mathcal{H}$ corresponding to u, v, respectively.

Theorem 5.8 is rephrased as follows:

Theorem 5.9. *Let B be a Boolean $(\bigvee, {}^\perp)$-sublattice of $P(\mathcal{H})$, and $u, v \in \mathfrak{R}_B$. Then for $p \in B$,*

$$p \leq [\![u =_{\mathbf{T}} v]\!]_B \iff [\![u \restriction p =_{\mathbf{2}} v \restriction p]\!]_B = 1,$$

where the **restriction** *$u \restriction p$ of u to p is defined by*

$$\mathcal{D}(u \restriction p) = \{\check{r} \mid r \in \mathbb{Q}\}, \quad (u \restriction p)(\check{r}) = u(\check{r}) \wedge p.$$

Proof.

$$\begin{aligned} p \leq [\![u =_{\mathbf{T}} v]\!]_B \quad &\iff \quad p \wedge u(\check{r}) = p \wedge v(\check{r}) \quad \text{for all } r \in \mathbb{Q} \\ &\iff \quad [\![u \restriction p =_{\mathbf{2}} v \restriction p]\!]_B = 1. \end{aligned}$$

□

Theorem 5.10. *Let B be a Boolean $(\bigvee, {}^\perp)$-sublattice of $P(\mathcal{H})$. If $p \in B$ and if u, v are real numbers in V^B, then there exists a real number w in V^B such that*

$$p \leq [\![u =_{\mathbf{T}} w]\!]_B \quad \textit{and} \quad p^\perp \leq [\![v =_{\mathbf{T}} w]\!]_B.$$

Proof. Let $\mathcal{D}w = \{\check{r} \mid r \in \mathbb{Q}\}, \quad w(\check{r}) = (u(\check{r}) \wedge p) \vee (v(\check{r}) \wedge p^\perp)$ □

Definition 5.11. Let $u, v \subset \check{\mathbb{Q}}$ be real numbers in $V^{P(\mathcal{H})}$. Then

$$u \leq_{\mathbf{2}} v \overset{\text{def}}{\iff} \forall r (r \in_{\mathbf{2}} v \rightarrow_{\mathbf{2}} r \in_{\mathbf{2}} u).$$

u is said to be **positive**, if $\check{0} \leq_{\mathbf{2}} u$.

$$u \leq_{\mathbf{T}} v \overset{\text{def}}{\iff} \forall r (r \in_{\mathbf{2}} v \rightarrow_{\mathbf{T}} r \in_{\mathbf{2}} u); \quad u <_{\mathbf{T}} v \overset{\text{def}}{\iff} (u \leq_{\mathbf{T}} v) \wedge (v \leq_{\mathbf{T}} u)^\perp$$

If $r, s \in \mathbb{Q}$, then $[\![\check{r} <_{\mathbf{T}} \check{s}]\!] = [\![\check{r} <_{\mathbf{2}} \check{s}]\!]$. So $\check{r} <_{\mathbf{T}} \check{s}$ may be written as $\check{r} < \check{s}$. We have

$$[\![\check{r} < \check{s}]\!] = 1 \iff r < s.$$

Definition 5.12. For compatible real numbers u, v in $V^{P(\mathcal{H})}$, $u + v$ is defined by

$$u + v \overset{\text{def}}{=} \{r \in_{\mathbf{2}} \check{\mathbb{Q}} \mid \forall s \in \check{\mathbb{Q}} (r < s \rightarrow_{\mathbf{T}} \exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}} (s =_{\mathbf{2}} s_1 + s_2 \ \wedge \ s_1 \in_{\mathbf{2}} u \ \wedge \ s_2 \in_{\mathbf{2}} v)\}.$$

For compatible positive real numbers u, v in $V^{P(H)}$, $u \cdot v$ is defined by

$$u \cdot v \overset{\text{def}}{=} \{r \in_{\mathbf{2}} \check{\mathbb{Q}} \mid \forall s \in_{\mathbf{2}} \check{\mathbb{Q}} (r < s \rightarrow_{\mathbf{T}} \exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}} (s =_{\mathbf{2}} s_1 \cdot s_2 \ \wedge \ s_1 \in_{\mathbf{2}} u \ \wedge \ s_2 \in_{\mathbf{2}} v)\}.$$

This definition of $u \cdot v$ is extended to the definition of $u \cdot v$ for all compatible real numbers u, v (cf. Takeuti[10]), and:

Theorem 5.13 (Takeuti [10]). *If u and v are mutually compatible real numbers in $V^{P(\mathcal{H})}$, corresponding to self-adjoint operators A_u, A_v, respectively,*

$$A_u = \int \lambda dE_u(\lambda) \qquad A_v = \int \lambda dE_v(\lambda),$$

then

$$A_{u+v} = \int \lambda dE_{u+v}(\lambda) = \int \lambda dE_u(\lambda) + \int \lambda dE_v(\lambda) = A_u + A_v$$

$$A_{u \cdot v} = \int \lambda dE_{u \cdot v}(\lambda) = \int \lambda dE_u(\lambda) \cdot \int \lambda dE_v(\lambda) = A_u \cdot A_v$$

Corollary 5.14. *If u is a real number in $V^{P(\mathcal{H})}$, i.e. $u \in \mathfrak{R}_{P(\mathcal{H})}$, then*

$$[\![u \cdot \check{1} =_{\mathbf{2}} u]\!]_{P(\mathcal{H})} = [\![u \cdot \check{0} =_{\mathbf{2}} \check{0}]\!]_{P(\mathcal{H})} = 1.$$

Definition 5.15. Let $\{q_j\}_{j \in J}$ be a complete orthogonal system, where $q_j = \sigma_U(p_j)$ for $j \in J$ and $U \in \mathcal{U}$. If $\{a_j\}_{j \in J} \subset \mathbb{R}$, then $\sum_{j \in J} \check{a}_j \cdot \widehat{q_j} \in V^{P(\mathcal{H})}$ is defined by

$$\mathcal{D}\left(\textstyle\sum_{j \in J} \check{a}_j \cdot \widehat{q_j}\right) = \{\check{r} \mid r \in \mathbb{Q}\}, \quad \left(\textstyle\sum_{j \in J} \check{a}_j \cdot \widehat{q_j}\right)(\check{r}) = \bigvee \{q_j \mid a_j \le r\}.$$

Then $\sum_{j \in J} \check{a}_j \cdot \widehat{q_j}$ is a real number in V^{B_U}.

Lemma 5.16. *Let $q_j = \sigma_U(p_j)$ for $j \in J$ and $U \in \mathcal{U}$. If $\{a_j\}_{j \in J} \subset \mathbb{R}$ and u is a real number in V^{B_U}, then*

$$q_j \le [\![u =_{\mathbf{T}} \check{a}_j]\!]_{P(\mathcal{H})} \quad \textit{for all } j \in J \iff [\![u =_{\mathbf{2}} \sum_{j \in J} \check{a}_j \cdot \widehat{q_j}]\!]_{P(\mathcal{H})} = 1.$$

Proof. For $r \in \mathbb{Q}$, if $a_j \le r$, then $q_j \wedge \check{a}_j(\check{r}) = q_j = q_j \wedge (\sum_{j \in J} \check{a}_j \cdot \widehat{q_j})(\check{r})$ and if $a_j > r$, then $q_j \wedge \check{a}_j(\check{r}) = 0 = q_j \wedge (\sum_{j \in J} \check{a}_j \cdot \widehat{q_j})(\check{r})$. Hence,

$$q_j \le [\![\check{a}_j =_{\mathbf{T}} \sum_{j \in J} \check{a}_j \cdot \widehat{q_j}]\!] \quad \text{for all } j \in J.$$

$$\begin{aligned}
\therefore \quad q_j \le [\![u =_{\mathbf{T}} \check{a}_j]\!]_{P(\mathcal{H})} \quad \text{for all } j \in J \quad &\iff \quad q_j \le [\![u =_{\mathbf{T}} \sum_{j \in J} \check{a}_j \cdot \widehat{q_j}]\!]_{P(\mathcal{H})} \quad \text{for all } j \in J \\
&\iff \quad [\![u =_{\mathbf{T}} \sum_{j \in J} \check{a}_j \cdot \widehat{q_j}]\!]_{P(\mathcal{H})} = 1 \\
&\iff \quad [\![u =_{\mathbf{2}} \sum_{j \in J} \check{a}_j \cdot \widehat{q_j}]\!]_{P(\mathcal{H})} = 1
\end{aligned}$$

□

Lemma 5.17. *Suppose that B is a Boolean $(\bigvee, {}^{\perp})$-sublattice of $P(\mathcal{H})$, u, v are real number in V^B, and $p \in B$. Then*

$$[\![u \restriction p + v \restriction p =_{\mathbf{2}} (u+v) \restriction p]\!]_{P(\mathcal{H})} = 1, \quad [\![u \restriction p \cdot v \restriction p =_{\mathbf{2}} (u \cdot v) \restriction p]\!]_{P(\mathcal{H})} = 1.$$

Proof.

$$\begin{aligned}
((u+v) \restriction p)(\check{r}) &= \left(\bigwedge_{s \in \mathbb{Q},\ r<s} [\![\exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}} (\check{s} =_{\mathbf{2}} s_1 + s_2 \ \wedge \ s_1 \in_{\mathbf{2}} u \wedge s_2 \in_{\mathbf{2}} v)]\!] \right) \wedge p \\
&= \left(\bigwedge_{s \in \mathbb{Q},\ r<s} \ \bigvee_{s_1, s_2 \in \mathbb{Q},\ s = s_1 + s_2} u(\check{s}_1) \wedge v(\check{s}_2)) \right) \wedge p \\
&= \bigwedge_{s \in \mathbb{Q},\ r<s} \ \bigvee_{s_1, s_2 \in \mathbb{Q},\ s = s_1 + s_2} (u \restriction p)(\check{s}_1) \wedge (v \restriction p)(\check{s}_2) \\
&= (u \restriction p + v \restriction p)(\check{r}).
\end{aligned}$$

$[\![u \restriction p \cdot v \restriction p =_{\mathbf{2}} (u \cdot v) \lceil p]\!]_{P(\mathcal{H})} = 1$ is proved similarly. □

Theorem 5.18 (Takeuti [10])**.** *Let $B \subset P(\mathcal{H})$ be a Boolean $(\bigvee, {}^{\perp})$-sublattice of $P(\mathcal{H})$. If u, v are real numbers in V^B, then $u+v$ and $u \cdot v$ are real numbers in V^B, and if u_1, u_2, v_1, v_2 are real numbers in V^B, then*

$$[\![u_1 =_{\mathbf{T}} u_2]\!]_{P(\mathcal{H})} \wedge [\![v_1 =_{\mathbf{T}} v_2]\!]_{P(\mathcal{H})} \leq [\![u_1 + v_1 =_{\mathbf{T}} u_2 + v_2]\!]_{P(\mathcal{H})} \wedge [\![u_1 \cdot v_1 =_{\mathbf{T}} u_2 \cdot v_2]\!]_{P(\mathcal{H})}.$$

Proof. If u_1, u_2, v_1, v_2 are real numbers in V^B, then $[\![u_1 =_{\mathbf{T}} u_2]\!]_{P(\mathcal{H})} = [\![u_1 =_{\mathbf{T}} u_2]\!]_B$ and $[\![v_1 =_{\mathbf{T}} v_2]\!]_{P(\mathcal{H})} = [\![v_1 =_{\mathbf{T}} v_2]\!]_B$. Let $[\![u_1 =_{\mathbf{T}} u_2]\!]_{P(\mathcal{H})} \wedge [\![v_1 =_{\mathbf{T}} v_2]\!]_{P(\mathcal{H})} = p$. Then

$$[\![u_1 \restriction p =_{\mathbf{2}} u_2 \restriction p]\!]_B \wedge [\![v_1 \restriction p =_{\mathbf{2}} v_2 \restriction p]\!]_B = 1.$$

By using Lemma 5.17,

$$[\![(u_1 + v_1) \restriction p =_{\mathbf{2}} (u_2 + v_2) \restriction p]\!]_B = 1.$$

Hence,

$$p \leq [\![u_1 + v_1 =_{\mathbf{T}} u_2 + v_2]\!]_B.$$

Similarly,

$$p \leq [\![u_1 \cdot v_1 =_{\mathbf{T}} u_2 \cdot v_2]\!]_B.$$

□

5.3 Complex numbers in $V^{P(\mathcal{H})}$

We can also treat complex numbers in $V^{P(\mathcal{H})}$. **Complex number** is defined as a pair $< u, v >$ of compatible real numbers u, v, which is denoted by $u + iv$. i denotes the complex number $< \check{0}, \check{1} >$, which is a check set associated with $< 0, 1 >$. The set of all complex numbers is denoted by $\mathfrak{C}$.

$$\mathfrak{C} \stackrel{\text{def}}{=} \{u + iv \mid u \in_{\mathbf{2}} \mathfrak{R} \wedge v \in_{\mathbf{2}} \mathfrak{R} \wedge u \downharpoonleft_{\circ} v\}.$$

The set of complex numbers in $V^{P(\mathcal{H})}$ is denoted by $\mathfrak{C}_{P(\mathcal{H})}$.

$$\mathfrak{C}_{P(\mathcal{H})} = \{u + iv \mid u \in \mathfrak{R}_{P(\mathcal{H})} \wedge v \in \mathfrak{R}_{P(\mathcal{H})} \wedge [\![u \downharpoonleft_{\circ} v]\!]_{P(\mathcal{H})} = 1\}.$$

Complex numbers $u = u_1 + iu_2$ and $v = v_1 + iv_2$ are said to be **compatible**, in symbols $u \downharpoonleft_{\circ} v$, if $\{u_1, u_2, v_1, v_2\}$ is compatible.

$$(u_1 + iu_2) \downharpoonleft_{\circ} (v_1 + iv_2) \stackrel{\text{def}}{\Longleftrightarrow} \downharpoonleft_{\circ} \{u_1, u_2, v_1, v_2\}$$

Since real numbers u_1, u_2 in V^B for some Boolean $(\bigvee, {}^{\perp})$-sublattice B of $P(\mathcal{H})$ represent compatible self-adjoint operators on $\mathcal{H}$, $u = u_1 + iu_2$ represents a normal operator on $\mathcal{H}$. Complex number representing normal operator A will be denoted by $\widehat{A}$.

5.4 Representation of propositions in $V^{P(\mathcal{H})}$

Propositions are represented by projections. A projection $p \in P(\mathcal{H})$ is a self-adjoint operator on $\mathcal{H}$, and represented by a real number $\widehat{p}$ in $V^{P(\mathcal{H})}$.

Theorem 5.19 (Titani [13]). *Let u be a real number in $V^{P(\mathcal{H})}$ and p be a projection on $\mathcal{H}$ i.e. $p \in P(\mathcal{H})$. Then*

$$[\![u =_{\mathbf{2}} \widehat{p}]\!]_{P(\mathcal{H})} = 1 \iff [\![u =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = [\![u =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = p.$$

Proof. Projection p is a self-adjoint operator with two eigen-values 0 and 1, hence

$$p = \int \lambda dE_{\widehat{p}}(\lambda) = 0 \cdot E_{\widehat{p}}(0) + 1 \cdot (E_{\widehat{p}}(1) - E_{\widehat{p}}(0)) = E_{\widehat{p}}(0)^{\perp},$$

$$E_{\widehat{p}}(1) = \bigwedge_{1<r} [\![\check{r} \in_{\mathbf{2}} \widehat{p}]\!] = 1, \quad \therefore\ E_{\widehat{p}}(0) = p^{\perp}.$$

It follows that

$$\widehat{p}(\check{r}) = \begin{cases} 1, & \text{if } 1 \leq r, \\ p^{\perp}, & \text{if } 0 \leq r < 1, \\ 0, & \text{if } r < 0. \end{cases}$$

It suffices to show that

$$u(\check{r}) = \begin{cases} 1, & \text{if } 1 \le r, \\ p^\perp, & \text{if } 0 \le r < 1, \\ 0, & \text{if } r < 0, \end{cases} \quad \Longleftrightarrow \quad [\![u =_{\mathbf{T}} \check{1}]\!] = [\![u =_{\mathbf{T}} \check{0}]\!]^\perp = p,$$

where $\check{1}$ and $\check{0}$ are check sets associated with real numbers 1 and 0, respectively, i.e. $\check{1} = \{r \in \mathbb{Q} \mid 1 \le r\}\check{}$, $\check{0} = \{r \in \mathbb{Q} \mid 0 \le r\}\check{}$.

Proof of $(\Rightarrow)$ **:** If $1 \le r$, then

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{1}]\!]) \wedge (\check{1}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = 1,$$

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{0}]\!]) \wedge (\check{0}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = 1;$$

If $0 \le r < 1$, then

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{1}]\!]) \wedge (\check{1}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = p,$$

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{0}]\!]) \wedge (\check{0}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = p^\perp;$$

If $r < 0$, then

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{1}]\!]) \wedge (\check{1}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = 1,$$

$$(u(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} \check{0}]\!]) \wedge (\check{0}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{T}} u]\!]) = 1.$$

Hence we have $[\![u =_{\mathbf{T}} \check{1}]\!] = [\![u =_{\mathbf{T}} \check{0}]\!]^\perp = p$.

Proof of $(\Leftarrow)$**:** If $1 \le r$, then

$$p \le (\check{1}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{2}} u]\!]) \le u(\check{r}), \quad p^\perp \le (\check{0}(\check{r}) \to_{\mathbf{T}} [\![\check{r} \in_{\mathbf{2}} u]\!]) \le u(\check{r})$$

It follows that $u(\check{r}) \ge p \vee p^\perp = 1$;

If $0 \le r < 1$, then $p \le u(\check{r})^\perp$ and $p^\perp \le u(\check{r})$. Hence, $u(\check{r}) = p^\perp$.

If $r < 0$, then $p \le u(\check{r})^\perp$ and $p^\perp \le u(\check{r})^\perp$. Hence, $u(\check{r}) \le p \wedge p^\perp = 0$.

□

Definition 5.20. A real number u is called a **proposition** if

$$u =_{\mathbf{T}} \check{1} \leftrightarrow_{\mathbf{2}} (u =_{\mathbf{T}} \check{0})^\perp.$$

The set of propositions is denoted by Prop.

$$\mathrm{Prop} \overset{\text{def}}{=} \{u \in \mathfrak{R} \mid u =_{\mathbf{T}} \check{1} \leftrightarrow_{\mathbf{2}} (u =_{\mathbf{T}} \check{0})^\perp\}$$

Propositions u, v are said to be **orthogonal** if $u =_{\mathbf{T}} \check{1} \to_{\mathbf{2}} (v =_{\mathbf{T}} \check{1})^\perp$.

$$u \perp v \overset{\text{def}}{\Longleftrightarrow} (u, v \in_{\mathbf{2}} \mathrm{Prop}) \wedge (u =_{\mathbf{T}} \check{1} \to_{\mathbf{2}} (v =_{\mathbf{T}} \check{1})^\perp).$$

Lemma 5.21. *For* $p, q \in P(\mathcal{H})$,

1. $p \leq q \Longleftrightarrow [\![\widehat{p} \leq_{\mathbf{2}} \widehat{q}]\!]_{P(\mathcal{H})} = 1$.
2. $p \perp q \Longleftrightarrow [\![\widehat{p} \perp \widehat{q}]\!]_{P(\mathcal{H})} = 1$.
3. $[\![\widehat{1} =_{\mathbf{2}} \check{1}]\!]_{P(\mathcal{H})} = [\![\widehat{0} =_{\mathbf{2}} \check{0}]\!]_{P(\mathcal{H})} = 1$.
4. $[\![\bigvee_{k\in K} \widehat{p}_k =_{\mathbf{2}} \widehat{\bigvee_{k\in K} p_k}]\!]_{P(\mathcal{H})} = [\![\bigwedge_{k\in K} \widehat{p}_k =_{\mathbf{2}} \widehat{\bigwedge_{k\in K} p_k}]\!]_{P(\mathcal{H})} = 1$.

Proof. 1. For $r \in \mathbb{Q}$, if $r < 0$ or $1 \leq r$, then $[\![\check{r} \in_{\mathbf{2}} \widehat{p}]\!]_{P(\mathcal{H})} = [\![\check{r} \in_{\mathbf{2}} \widehat{q}]\!]_{P(\mathcal{H})}$ is obvious, and if $0 \leq r < 1$, then $p \leq q \Longleftrightarrow [\![\check{r} \in_{\mathbf{2}} \widehat{q}]\!]_{P(\mathcal{H})} = q^{\perp} \leq p^{\perp} = [\![\check{r} \in_{\mathbf{2}} \widehat{p}]\!]_{P(\mathcal{H})}$.

2. $p \perp q \Longleftrightarrow p \leq q^{\perp} \Longleftrightarrow [\![\widehat{p} =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} \leq [\![\widehat{q} =_{\mathbf{T}} \check{1}]\!]^{\perp}_{P(\mathcal{H})}$.

3. By Theorem 5.19, $[\![\widehat{p} =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = [\![\widehat{p} =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = p$.

4. By 1.

□

Definition 5.22. A minimal proposition is called an **atom**, and the set of all atoms is denoted by Atom.

$$\text{Atom} \overset{\text{def}}{=} \{a \in_{\mathbf{2}} \text{Prop} \mid \neg_{\mathbf{2}}(a =_{\mathbf{2}} \check{0}) \wedge \forall x \in \text{Prop}(x \leq_{\mathbf{2}} a \rightarrow_{\mathbf{2}} (x =_{\mathbf{2}} \check{0} \vee x =_{\mathbf{2}} a)\}.$$

A set A of mutually orthogonal atoms with $\bigvee A =_{\mathbf{2}} \check{1}$ is called a **complete orthogonal system**, i.e.

$$(A \subset_{\mathbf{2}} \text{Atom}) \wedge (\forall x, y \in_{\mathbf{2}} A(x \perp y)) \wedge (\bigvee A =_{\mathbf{2}} \check{1}).$$

Lemma 5.23. *1.* a *is an atom* $\Longleftrightarrow [\![\widehat{a} \in_{\mathbf{2}} \text{Atom}]\!]_{P(\mathcal{H})} = 1$, *for* $a \in P(\mathcal{H})$.

2. $\{p_j\}_{j\in J}$ *is a complete orthogonal system* $\Longleftrightarrow$
$[\![\{\widehat{p}_j\}_{j\in J}$ *is a complete orthogonal system*$]\!]_{P(\mathcal{H})} = 1$.

Proof. 1. $a \neq 0 \Longleftrightarrow a = [\![\widehat{a} =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} \leq [\![\widehat{a} =_{\mathbf{2}} \widehat{0}]\!]^{\perp}_{P(\mathcal{H})} = \neg_{\mathbf{2}}[\![\widehat{a} =_{\mathbf{2}} \widehat{0}]\!]_{P(\mathcal{H})} = 1$. Hence,

$$a \text{ is an atom} \Longleftrightarrow [\![\neg_{\mathbf{2}}(\widehat{a} =_{\mathbf{2}} \check{0}) \wedge \forall x \in \text{Prop}(x \leq_{\mathbf{2}} \widehat{a} \rightarrow_{\mathbf{2}} (x =_{\mathbf{2}} \check{0} \vee x =_{\mathbf{2}} \widehat{a})]\!]_{P(\mathcal{H})} = 1$$

.

2. By Lemma 5.21.

□

Theorem 5.24. $[\![\text{Prop}$ *is a proposition system* $]\!]_{P(\mathcal{H})} = 1$, *where* $\widehat{p}^{\perp} \overset{def}{\Longleftrightarrow} \widehat{p^{\perp}}$.

Proof. By Theorem 5.19, $p \in P(\mathcal{H}) \Longleftrightarrow [\![\widehat{p} \in_{\mathbf{2}} \text{Prop}]\!]_{P(\mathcal{H})} = 1$. Hence we have the theorem by Lemma 5.21. □

5.5 Structure of $\check{\mathbb{R}}$ and $\check{\mathbb{C}}$ in $V^{P(\mathcal{H})}$

For each $U \in \mathcal{U}$, B_U is a Boolean $(\bigvee, {}^{\perp})$-sublattice of $P(\mathcal{H})$ generated by complete orthogonal system $P_U = \{\sigma_U(p_j)\}_{j\in J}$, and $P(\mathcal{H}) = \bigcup_{U\in\mathcal{U}} B_U$. In this section, we prove that

$$\begin{aligned}
[\![u \in_{\mathbf{T}} \check{\mathbb{R}}]\!]_{P(\mathcal{H})} = 1 \quad &\Longleftrightarrow \quad \exists U \in \mathcal{U}\, \exists \{a_j\} \subset \mathbb{R}\,([\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1),\\
&\Longleftrightarrow \quad \exists U \in \mathcal{U}(u \in \mathfrak{R}_{B_U})
\end{aligned}$$

$$\begin{aligned}
[\![u \in_{\mathbf{T}} \check{\mathbb{C}}]\!]_{P(\mathcal{H})} = 1 \quad &\Longleftrightarrow \quad \exists U \in \mathcal{U}\, \exists \{a_j\} \subset \mathbb{C}\,([\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1),\\
&\Longleftrightarrow \quad \exists U \in \mathcal{U}(u \in \mathfrak{C}_{B_U})
\end{aligned}$$

Lemma 5.25. *Let B be a complete Boolean lattice, u be a real number in V^B, and $p \in B$. Then*

$$p \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} u]\!]_B, \quad \textit{and} \quad p^{\perp} \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} \check{0}]\!]_B.$$

Proof. By Theorem 5.18 and Corollary 5.14,

$$p = [\![u =_{\mathbf{T}} u]\!]_B \wedge [\![\widehat{p} =_{\mathbf{T}} \check{1}]\!]_B \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} u \cdot \check{1}]\!]_B \quad \text{and} \quad [\![u \cdot \check{1} =_{\mathbf{2}} u]\!]_B = 1.$$

Therefore, $p \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} u]\!]_B$.

$$p^{\perp} = [\![u =_{\mathbf{T}} u]\!]_B \wedge [\![\widehat{p} =_{\mathbf{T}} \check{0}]\!]_B \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} u \cdot \check{0}]\!]_B \quad \text{and} \quad [\![u \cdot \check{0} =_{\mathbf{2}} \check{0}]\!]_B = 1.$$

Therefore, $p^{\perp} \leq [\![u \cdot \widehat{p} =_{\mathbf{T}} \check{0}]\!]_B$. □

Lemma 5.26. *If u is a real number in $V^{P(\mathcal{H})}$ and $a, b \in \mathbb{R}$, then*

$$[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})} \wedge [\![u =_{\mathbf{T}} \check{b}]\!]_{P(\mathcal{H})} \leq [\![\check{a} =_{\mathbf{2}} \check{b}]\!]_{P(\mathcal{H})}.$$

Proof. If u is a real number in $V^{P(\mathcal{H})}$ and $a, b \in \mathbb{R}$, then, by Theorem 5.4, there exists a Boolean $(\bigvee, {}^{\perp})$-sublattice B such that $u \in V^B$. Since $\check{a}, \check{b}$ are in V^B, by equality axiom in V^B,

$$[\![u =_{\mathbf{T}} \check{a}]\!]_B \wedge [\![u =_{\mathbf{T}} \check{b}]\!]_B \leq [\![\check{a} =_{\mathbf{T}} \check{b}]\!]_B = [\![\check{a} =_{\mathbf{2}} \check{b}]\!]_B.$$

□

Theorem 5.27. *Let u be a real number in $V^{P(\mathcal{H})}$. Then*

$$[\![u \in_{\mathbf{T}} \check{\mathbb{R}}]\!]_{P(\mathcal{H})} = 1 \Longleftrightarrow \exists U \in \mathcal{U}\, \exists \{a_j\}_{j\in J} \subset \mathbb{R}\,(\, [\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1\,).$$

Proof. If u is a real number in $V^{P(\mathcal{H})}$ and $[\![u \in_{\mathbf{T}} \check{\mathbb{R}}]\!]_{P(\mathcal{H})} = 1$, then $\bigvee_{a\in\mathbb{R}}[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})} = 1$. If $a, b \in \mathbb{R}$ and $a \neq b$, then $[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})} \wedge [\![u =_{\mathbf{T}} \check{b}]\!]_{P(\mathcal{H})} = 0$ by Lemma 5.26, hence, $[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})} \perp [\![u =_{\mathbf{T}} \check{b}]\!]_{P(\mathcal{H})}$.

It follows that $\{[\![u =_{\mathbf{T}} \check{x}]\!]_{P(\mathcal{H})} \mid x \in \mathbb{R}\}$ is an orthogonal subset of $P(\mathcal{H})$. Each $[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})}$ is a supremum of atoms, i.e. $[\![u =_{\mathbf{T}} \check{a}]\!]_{P(\mathcal{H})} = \bigvee_{j\in J_a} q_j$. Let $J = \bigcup_{a\in\mathbb{R}} J_a$. $\{q_j\}_{j\in J} = \bigcup_{a\in\mathbb{R}}\{q_j \mid j \in J_a\}$ is a complete orthogonal system. Then, there exists $U \in \mathcal{U}$ such that $q_j = \sigma_U(p_j)$ for each $j \in J$.

For each $j \in J_a$, let $a_j = a$. We have

$$q_j \leq [\![u =_{\mathbf{T}} \check{a}_j]\!]_{P(\mathcal{H})} \quad \text{and} \quad q_j = [\![\check{1} =_{\mathbf{T}} \widehat{q}_j]\!]_{P(\mathcal{H})} = [\![\check{0} =_{\mathbf{T}} \widehat{q}_j]\!]^{\perp}_{P(\mathcal{H})}.$$

Therefore, by Lemma 5.16, $[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1$.

Conversely if $[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1$ for $U \in \mathcal{U}$ and $\{a_j\}_{j\in J} \subset \mathbb{R}$, then $\sigma_U(p_j) \leq [\![u =_{\mathbf{T}} \check{a}_j \in \check{\mathbb{R}}]\!]_{P(\mathcal{H})}$ for each $j \in J$. Therefore, $[\![u \in_{\mathbf{T}} \check{\mathbb{R}}]\!]_{P(\mathcal{H})} = 1$. □

Theorem 5.28. *If u and v are real numbers in $V^{P(\mathcal{H})}$ such that*

$$[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1, \quad [\![v =_{\mathbf{2}} \sum_{j\in J} \check{b}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1,$$

for some complete orthogonal system $\{q_j\}_{j\in J}$ and $\{a_j\}_{j\in J}$, $\{b_j\}_{j\in J} \subset \mathbb{R}$, then

$$[\![u + v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j + \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1 \quad \textit{and} \quad [\![u \cdot v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j \cdot \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1.$$

Proof. Let B be the Boolean $(\bigvee, ^{\perp})$-sublattice of $P(\mathcal{H})$ generated by $\{q_j\}_{j\in J}$. Then u, v are real numbers in V^B. By Theorem 5.18,

$$q_j \leq [\![u =_{\mathbf{T}} \check{a}_j]\!]_B \wedge [\![v =_{\mathbf{T}} \check{b}_j]\!]_B \leq [\![u + v =_{\mathbf{T}} \check{a}_j + \check{b}_j]\!]_B \quad \text{for all } j \in J.$$

By Lemma 5.16,

$$[\![u + v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j + \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1.$$

Similarly,

$$[\![u \cdot v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j \cdot \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1.$$

□

Since the operator σ_U on $P(\mathcal{H})$ defined by $\sigma_U(p) = UpU^*$ is a bijection preserving $\bigvee$ and $^{\perp}$, σ_U is extended to the isomorphism between universes $\sigma_U : V^{P(\mathcal{H})} \to V^{P(\mathcal{H})}$ (Definition 4.4). Then we have:

Lemma 5.29.

$$[\![\widehat{\sigma_U(p)} =_{\mathbf{2}} \sigma_U(\widehat{p})]\!]_{P(\mathcal{H})} = 1 \quad \textit{for } U \in \mathcal{U} \textit{ and } p \in P(\mathcal{H}).$$

Proof. By Theorem 5.19, $[\![\widehat{p} =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = [\![\widehat{p} =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = p$. Hence, by Theorem 2.36,

$$\begin{aligned} \sigma_U([\![\widehat{p} =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})}) &= [\![\sigma_U(\widehat{p}) =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = \sigma_U(p) \\ \sigma_U([\![\widehat{p} =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})}) &= [\![\sigma_U(\widehat{p}) =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = \sigma_U(p). \end{aligned}$$

That is, $\sigma_U(\widehat{p})$ is a projection and

$$[\![\sigma_U(\widehat{p}) =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = [\![\sigma_U(\widehat{p}) =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = \sigma_U(p).$$

By the uniqueness of $\widehat{\sigma_U(p)}$ such that

$$[\![\widehat{\sigma_U(p)} =_{\mathbf{T}} \check{1}]\!]_{P(\mathcal{H})} = [\![\widehat{\sigma_U(p)} =_{\mathbf{T}} \check{0}]\!]^{\perp}_{P(\mathcal{H})} = \sigma_U(p),$$

$[\![\widehat{\sigma_U(p)} =_{\mathbf{2}} \sigma_U(\widehat{p})]\!]_{P(\mathcal{H})} = 1$ (cf. Theorem 5.19). □

From Theorem 5.27 and Theorem 5.28, we have:

Corollary 5.30. *1. For a complex number u in $V^{P(\mathcal{H})}$, $[\![u \in_{\mathbf{T}} \check{\mathbb{C}}]\!]_{P(\mathcal{H})} = 1$ if and only if there exists $U \in \mathcal{U}$ and $\{a_j\}_{j\in J} \subset \mathbb{C}$ such that*

$$[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1.$$

2. If $u, v \in V^{P(\mathcal{H})}$ satisfy $[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = [\![v =_{\mathbf{2}} \sum_{j\in J} \check{b}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1$ for some complete orthogonal system $\{q_j\}_{j\in J}$ and $\{a_j\}_{j\in J}$, $\{b_j\}_{j\in J} \subset \mathbb{C}$, then

$$[\![u + v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j + \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1 \quad \textit{and} \quad [\![u \cdot v =_{\mathbf{2}} \sum_{j\in J} (\check{a}_j \cdot \check{b}_j) \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1.$$

Theorem 5.31. *If u is a real number in V^{B_U} for $U \in \mathcal{U}$, then there exists $\{a_j\}_{j\in J} \subset \mathbb{R}$ such that $[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1$.*

Proof. For each $j \in J$, set

$$a_j = \inf\{r \in \mathbb{Q} \mid \sigma_U(p_j) \le u(\check{r})\}.$$

Since $\sigma_U(p_j)$ is an atom in B_U,

$$\sigma_U(p_j) \wedge u(\check{r}) = 0 \quad \text{or} \quad \sigma_U(p_j) \le u(\check{r}) \quad \text{for each } r \in \mathbb{Q}.$$

$$\therefore \quad \sigma_U(p_j) \wedge u(\check{r}) = \sigma_U(p_j) \wedge \check{a}_j(\check{r}).$$

It follows that $\sigma_U(p_j) \le [\![u =_{\mathbf{T}} \check{a}_j]\!]_{P(\mathcal{H})}$ for all $j \in J$. Hence, by Lemma 5.16,

$$[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!] = 1.$$

□

Corollary 5.32. *If $w = u + iv$ is a complex number in V^{B_U} for $U \in \mathcal{U}$, then there exists $\{c_j\}_{j\in J} \subset \mathbb{C}$ such that $[\![w =_{\mathbf{2}} \sum_{j\in J} \check{c}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1$.*

Proof. By Theorem 5.31, there exist $\{a_j\}_{j\in J}$, $\{b_j\}_{j\in J} \subset \mathbb{R}$ such that

$$[\![u =_{\mathbf{2}} \sum_{j\in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = [\![v =_{\mathbf{2}} \sum_{j\in J} \check{b}_j \cdot \widehat{\sigma_U(p_j)}]\!]_{P(\mathcal{H})} = 1.$$

Let $c_j = a_j + ib_j$ for each $j \in J$. □

5.6 Representation of $\mathcal{H}$ in $V^{P(\mathcal{H})}$

Vector x in $\mathcal{H}$ has expression as a linear combination of basis $\{Ue_j\}_{j\in J}$, for each $U \in \mathcal{U}$:

$$x = \sum_{j\in J} (Ue_i, x) \cdot Ue_j.$$

Each expression is represented by a normal operator on $\mathcal{H}$:

$$\sum_{j\in J} (Ue_i, x) \cdot \sigma_U(p_j),$$

where (,) denotes the inner product in $\mathcal{H}$. The normal operator is represented by complex number $\widehat{x}_U$ in $V^{P(\mathcal{H})}$:

$$\widehat{x}_U = \sum_{j\in I} (Ue_j, x)\check{} \cdot \widehat{\sigma_U(p_j)} \in \mathfrak{C}_{B_U}.$$

The coordinates of $\widehat{x}_U$ with respect to bases $\{\widehat{\sigma_U(p_j)}\}_{j\in J}$ are check sets and represented by matrix:

$$\begin{bmatrix} \vdots \\ (Ue_j, x)\check{} \\ \vdots \end{bmatrix} \in \check{\mathbb{C}}^d,$$

where d is the dimension of $\mathcal{H}$. Since $V^{\mathbf{2}} \cong V$, the matrix is represented as a matrix in V, which is denoted by $[\widehat{x}_U]$:

$$[\widehat{x}_U] = \begin{bmatrix} \vdots \\ (Ue_j, x) \\ \vdots \end{bmatrix} \in \mathbb{C}^d.$$

$\{< U, [\widehat{x}_U] >|\ U \in \mathcal{U},\ x \in \mathcal{H}\}$ forms a sheaf $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ over $\mathcal{U}$ with stalks isomorphic to $\mathcal{H}$, That is,

$$\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}) = < \widetilde{\mathcal{H}}, \pi >, \quad \text{where}$$

$$\widetilde{\mathcal{H}} = \{< U,\ [\widehat{x}_U] >|\ U \in \mathcal{U},\ x \in \mathcal{H}\},$$

$$\pi : \mathrm{Sh}_{\mathcal{U}}(\mathcal{H}) \to \mathcal{U} \ \text{ defined by } \ < U,\ |x\rangle_U > \mapsto U.$$

$\mathcal{H}$ is represented by the sheaf $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ of Hilbert space, in $V^{P(\mathcal{H})}$.

Remark 5.33. Let $U_0 = e^{\frac{\pi}{2}iE} = iE$, where E is the identity on $\mathcal{H}$, and let $U \in \mathcal{U}$. For a complete orthogonal system $\{\sigma_U(p_j)\}_{j\in J} \subset B_U$,

$$\sigma_{U_0}(\sigma_U(p_j)) = \sigma_U(p_j) \quad \text{for } j \in J. \quad \text{Hence, } B_U = B_{U_0U}.$$

Then vector $x = \sum_{j\in J}(Ue_i, x) \cdot Ue_j$ in $\mathcal{H}$ is represented as complex number in both of V^{B_U} and $V^{B_{U_0U}}$:

$$\widehat{x}_U = \sum_{j\in I}(Ue_j, x)\check{} \cdot \widehat{q_j} \in \mathfrak{C}_{B_U},$$

$$\widehat{x}_{U_0U} = \sum_{j\in I}(U_0(Ue_j), x)\check{} \cdot \sigma_{U_0}\widehat{(\sigma_U(p_j))} = \sum_{j\in I}(iUe_j, x)\check{} \cdot \widehat{\sigma_U(p_j)} \in \mathfrak{C}_{B_{U_0U}}.$$

Hence,

$$[\widehat{x}_{UU_0}] = i[\widehat{x}_U].$$

That is, a vector $x \in \mathcal{H}$ is represented in $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ by $[\widehat{x}_U]$ in the stalk on U, and also by $i[\widehat{x}_U]$ in the stalk on U_0U.

5.7 R-, L-, RL-complex numbers of $V^{\mathcal{Q}}$

By Theorem 2.35, we have:

Theorem 5.34. *1. If $\varphi(u_1, \cdots u_n)$ is a bounded formula of quantum set theory* QZFZ *with constants $u_1, \cdots u_n$ in $V^{P(\mathcal{H})}$, then*

$$\chi_R([\![\varphi(u_1, \cdots u_n)]\!]_{P(\mathcal{H})}) = [\![\varphi(\chi_R(u_1), \cdots \chi_R(u_n))]\!]_{R(\mathcal{Q})}$$

$$\chi_L([\![\varphi(u_1, \cdots u_n)]\!]_{P(\mathcal{H})}) = [\![\varphi(\chi_L(u_1), \cdots \chi_L(u_n))]\!]_{L(\mathcal{Q})}$$

If constants $u_1, \cdots u_n$ are in V^{B_U} for $U \in \mathcal{U}$, then

$$\chi^U_{RL}([\![\varphi(u_1, \cdots u_n)]\!]_{B_U}) = [\![\varphi(\chi^U_{RL}(u_1), \cdots \chi_{RL}(u_n))^U]\!]_{\chi^U_{RL}(B_U)}.$$

2. If $\varphi(u_1, \cdots u_n)$ is a formula of QZFZ *with constants $u_1, \cdots u_n$ in $V^{R(\mathcal{Q})}$, then*

$$[\![\varphi(u_1, \cdots u_n)]\!]^*_{R(\mathcal{Q})} = [\![\varphi(u_1^*, \cdots u_n^*)]\!]_{L(\mathcal{Q})}$$

Corollary 5.35. *Let f be χ_R, χ_L or χ^U_{RL} for $U \in \mathcal{U}$.*

1. If $u, v \in V^{B_U}$, $f[\![\,u$ is a real number $]\!]_{B_U} = [\![f(u)$ is a real number $]\!]_{f(B_U)}$,

and $f[\![\,u$ is a complex number $]\!]_{B_U} = [\![f(u)$ is a complex number $]\!]_{f(B_U)}$.

2. *If* u, v *are complex numbers in* V^{B_U}*, then*

$$[\![f(u+v) =_{\mathbf{2}} f(u) + f(v)]\!]_{f(B_U)} = 1$$

3. *If* u_1, u_2, v_1, v_2 *are real numbers in* V^{B_U}*, then by Theorem 4.19,*

$$\begin{aligned}
&[\![f(u_1) =_{\mathbf{r}} f(v_1)]\!]_{f(B_U)} \wedge [\![f(u_2) =_{\mathbf{r}} f(v_2)]\!]_{f(B_U)} \\
&\qquad\qquad \le [\![f(u_1) + f(v_1) =_{\mathbf{r}} f(u_2) + f(v_2)]\!]_{f(B_U)}, \\
&[\![f(u_1) =_{\mathbf{r}} f(v_1)]\!]_{f(B_U)} \wedge [\![f(u_2) =_{\mathbf{r}} f(v_2)]\!]_{f(B_U)} \\
&\qquad\qquad \le [\![f(u_1) \cdot f(v_1) =_{\mathbf{r}} f(u_2) \cdot f(v_2)]\!]_{f(B_U)}.
\end{aligned}$$

4. $[\![f(\widehat{q_j}) =_{\mathbf{2}} \widehat{f(q_j)}]\!]_{f(B_U)} = 1$*, where* $q_j = \sigma_U(p_j)$

5. $[\![f(\sum_{j\in J} \breve{a}_j \cdot \widehat{q_j}) =_{\mathbf{2}} \sum_{j\in J} \breve{a}_j \cdot \widehat{f(q_j)}]\!]_{f(B_U)} = 1$*, for* $\{a_j\}_{j\in J} \subset \mathbb{C}$ *and* $q_j = \sigma_U(p_j)$*.*

Definition 5.36. If u is a complex number in $V^{P(\mathcal{H})}$, then $\chi_R(u)$ is a complex number in $V^{R(\mathcal{Q})}$, and $\chi_L(u)$ is a complex number in $V^{L(\mathcal{Q})}$.

$$\chi_R(u) \in \mathfrak{C}_{R(\mathcal{Q})}, \qquad \chi_L(u) \in \mathfrak{C}_{L(\mathcal{Q})}.$$

A complex number in $V^{R(\mathcal{Q})}$ is called a **right-complex** and a complex number in $V^{L(\mathcal{Q})}$ is called a **left-complex**.

$$u + iv \text{ is a left-complex} \iff (u+iv)^* = (u^* - iv^*) \text{ is a right-complex.}$$

In what follows, we write $\chi_R(u)$ as $(u)_R$ and $\chi_L(u)$ as $(u)_L$. If $U \in \mathcal{U}$, and u is a complex number in V^{B_U}, then $\chi_{RL}^U(u)$ in $V^{\chi_{RL}^U(B_U)}$ is denoted by $(u)_{RL}^U$.

$$(u)_R \stackrel{\text{def}}{=} \chi_R(u), \quad (u)_L \stackrel{\text{def}}{=} \chi_L(u) \quad \text{for } u \in \mathfrak{C}_{P(\mathcal{H})}$$

$$(u)_{RL}^U \stackrel{\text{def}}{=} \chi_{RL}^U(u), \quad \text{for } u \in \mathfrak{C}_{B_U}$$

Since

$$\chi_R : V^{B_U} \cong V^{\chi_R(B_U)}, \quad \chi_L : V^{B_U} \cong V^{\chi_L(B_U)}, \quad \chi_{RL}^U : V^{B_U} \cong V^{\chi_{RL}^U(B_U)},$$

$$\begin{aligned}
u \text{ is a complex number in } V^{B_U} &\iff (u)_{RL}^U \text{ is a complex number in } V^{\chi_{RL}^U(B_U)} \\
&\iff (u)_R \text{ is a complex number in } V^{\chi_R(B_U)} \\
&\iff (u)_L \text{ is a complex number in } V^{\chi_L(B_U)}.
\end{aligned}$$

If $U \in \mathcal{U}$ and $p, q \in B_U$, then $\chi_{RL}^U(p) \& \chi_{RL}^U(q)$, $\chi_{RL}^U(p) \& \chi_R^U(q)$, $\chi_L^U(p) \& \chi_{RL}^U(q)$ and $\chi_{LR}^U(p) \& \chi_{LR}^U(q)$ are defined, and by Theorem 3.21 and Theorem 3.24,

$$\begin{aligned}
\chi_{RL}^U(p) \& \chi_{RL}^U(q) &= \chi_{RL}^U(p) \wedge \chi_{RL}^U(q) = \chi_{RL}^U(p \wedge q), \\
\chi_{RL}^U(p) \& \chi_R^U(q) &= \chi_R^U(p \wedge q), \\
\chi_L^U(p) \& \chi_{RL}^U(q) &= \chi_L^U(p \wedge q), \\
\chi_{LR}^U(p) \& \chi_{LR}^U(q) &= \chi_{LR}^U(p) \wedge \chi_{LR}^U(q).
\end{aligned}$$

For positive real numbers u, v in V^{B_U}, where $U \in \mathcal{U}$, the definition:

$$u \cdot v \overset{\text{def}}{=} \{r \in \check{\mathbb{Q}} \mid \forall s \in_{\mathbf{2}} \check{\mathbb{Q}}(r < s \rightarrow_{\mathbf{T}}$$
$$\exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}}((s =_{\mathbf{2}} s_1 \cdot s_2) \wedge (s_1 \in_{\mathbf{2}} u) \wedge s_2 \in_{\mathbf{2}} v))\}$$

is extended to $V^{\mathcal{Q}}$ as follows:

Definition 5.37. Let $\alpha, \beta, \gamma \in \{R,\ L, RL,\ LR\}$. If u, v be positive real numbers in V^{B_U} for $U \in \mathcal{U}$, and $\chi_\alpha^U(p) \& \chi_\beta^U(q)$ is defined for each $p, q \in B_U$, then $(u)_\alpha \& (v)_\beta$ is defined by

$$(u)_\alpha \& (v)_\beta \overset{\text{def}}{=} \{r \in_{\mathbf{2}} \check{\mathbb{Q}} \mid \forall s \in_{\mathbf{2}} \check{\mathbb{Q}}(r < s \rightarrow_{\mathbf{T}}$$
$$\exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}}((s =_{\mathbf{2}} s_1 \cdot s_2) \wedge (s_1 \in_{\mathbf{2}} (u)_\alpha \& s_2 \in_{\mathbf{2}} (v)_\beta)))\}.$$

Theorem 5.38. *Let $\chi_\alpha, \chi_\beta, \chi_\gamma \in \{\chi_R,\ \chi_L,\ \chi_{RL}^U,\ \chi_{LR}^U\}$ for $U \in \mathcal{U}$. If*

$$\chi_\alpha(p) \& \chi_\beta(q) = \chi_\gamma(p \wedge q) \quad \textit{for each } p, q \in B_U$$

then, for positive real numbers u, v in V^{B_U},

$$[\![(u)_\alpha \& (v)_\beta =_{\mathbf{2}} (u \cdot v)_\gamma]\!]_{\chi_\gamma(B_U)} = 1.$$

Hence,

$$\begin{aligned} [\![(u)_{RL}^U \& (v)_{RL}^U &=_{\mathbf{2}} (u \cdot v)_{RL}^U]\!]_{\chi_{RL}^U(B_U)} = 1, \\ [\![(u)_{RL}^U \& (v)_R &=_{\mathbf{2}} (u \cdot v)_R]\!]_{\chi_R(B_U)} = 1, \\ [\![(u)_L \& (v)_{RL}^U &=_{\mathbf{2}} (u \cdot v)_L]\!]_{\chi_L(B_U)} = 1. \end{aligned}$$

Proof. For $s_1,\ s_2 \in \mathbb{Q}$,

$$\begin{aligned} [\![\check{s}_1 \in_{\mathbf{2}} (u)_\alpha \ \&\ \check{s}_2 \in_{\mathbf{2}} (v)_\beta]\!]_{\chi_\gamma(B_U)} &= \chi_\alpha [\![\check{s}_1 \in_{\mathbf{2}} u]\!]_{B_U} \ \&\ \chi_\beta [\![\check{s}_2 \in_{\mathbf{2}} v]\!]_{B_U} \\ &= \chi_\gamma [\![\check{s}_1 \in_{\mathbf{2}} u \ \wedge\ \check{s}_2 \in_{\mathbf{2}} v]\!]_{B_U}. \end{aligned}$$

Hence,

$$\begin{aligned} &[\![\check{r} \in_{\mathbf{2}} ((u)_\alpha \& (v)_\beta)]\!]_{\chi_\gamma(B_U)} \\ &= [\![\forall s \in_{\mathbf{2}} \check{\mathbb{Q}}(r < s \rightarrow_{\mathbf{T}} \exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}}(s =_{\mathbf{2}} s_1 \cdot s_2 \wedge (s_1 \in_{\mathbf{2}} (u)_\alpha \ \&\ s_2 \in_{\mathbf{2}} (v)_\beta))]\!]_{\chi_\gamma(B_U)} \\ &= \chi_\gamma [\![\forall s \in_{\mathbf{2}} \check{\mathbb{Q}}(r < s \rightarrow_{\mathbf{T}} \exists s_1, s_2 \in_{\mathbf{2}} \check{\mathbb{Q}}(s =_{\mathbf{2}} s_1 \cdot s_2 \wedge s_1 \in_{\mathbf{2}} u \wedge s_2 \in_{\mathbf{2}} v))]\!]_{B_U} \\ &= \chi_\gamma [\![\check{r} \in_{\mathbf{2}} (u \cdot v)]\!]_{B_U} = [\![\check{r} \in_{\mathbf{2}} (u \cdot v)_\gamma]\!]_{\chi_\gamma(B_U)} \end{aligned}$$

□

If $U \in \mathcal{U}$, $q_j = \sigma_U(p_j)$ for $j \in J$, and u is a complex number in V^{B_U}, then there exists $\{a_j\}_{j \in J} \subset \mathbb{C}$ such that

$$[\![u =_{\mathbf{2}} \sum_{j \in_{\mathbf{2}} \check{J}} \check{a}_j \cdot \widehat{q_j}]\!]_{B_U} = [\![u =_{\mathbf{2}} \sum_{j \in_{\mathbf{2}} \check{J}} \check{a}_j \cdot \widehat{q_j}]\!]_{P(\mathcal{H})} = 1,$$

by Corollary 5.32. In what follows we denote the check set $\check{J}$ by J, for simplicity.

Lemma 5.39. *Let $U \in \mathcal{U}$ and $q_j = \sigma_U(p_j)$ for each $j \in J$. If $u \in V^{B_U}$ and $\{a_j\}_{j\in J} \subset \mathbb{C}$, then*

$$
\begin{aligned}
[\![u =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \check{a}_j \cdot \widehat{q}_j]\!]_{B_U} = 1 \quad &\Longleftrightarrow \quad [\![u =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \check{a}_j \cdot \widehat{q}_j]\!]_{P(\mathcal{H})} = 1 \\
&\Longleftrightarrow \quad [\![\chi_R(u) =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \check{a}_j \cdot (\widehat{q}_j)_R]\!]_{R(\mathcal{Q})} = 1 \\
&\Longleftrightarrow \quad [\![\chi_L(u) =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \breve{\check{a}}_j \cdot (\widehat{q}_j)_L]\!]_{L(\mathcal{Q})} = 1 \\
&\Longleftrightarrow \quad [\![\chi^U_{RL}(u) =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \breve{\check{a}}_j \cdot (\widehat{q}_j)_{RL}]\!]_{\chi^U_{RL}(B_U)} = 1
\end{aligned}
$$

Proof. By Thenrem 2.36 and by $[\![(\check{a})_{\chi_\alpha} =_{\mathbf{2}} \check{a}]\!]_{\chi_\alpha(B_U)} = 1$. □

Theorem 5.40. *Let $U \in \mathcal{U}$ and $q_j = \sigma_U(p_j)$ for each $j \in J$. If $u = \sum_{j\in_{\mathbf{2}} J} \check{a}_j \cdot \widehat{q}_j$ and $v = \sum_{j\in_{\mathbf{2}} J} \check{b}_j \cdot \widehat{q}_j$, where $a_j, b_j \in \mathbb{C}$ for each $j \in J$, then*

$$
[\![(u)^U_{RL}\&(u)_R =_{\mathbf{2}} \sum_{i,j\in_{\mathbf{2}} J} \check{a}_i\check{b}_j \cdot (\widehat{q}_i)^U_{RL}\&(\widehat{q}_j)_R =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \check{a}_j\check{b}_j \cdot (\widehat{q}_j)_R]\!]_{\chi_R(B_U)} = 1 \tag{5.3}
$$

$$
[\![(u)_L\&(v)^U_{RL} =_{\mathbf{2}} \sum_{i,j\in_{\mathbf{2}} J} \breve{\check{a}}_i\check{b}_j \cdot (\widehat{q}_i)_L\&(\widehat{q}_j)^U_{RL} =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \breve{\check{a}}_j\check{b}_j \cdot (\widehat{q}_j)_L]\!]_{\chi_L(B_U)} = 1 \tag{5.4}
$$

$$
[\![(u)^U_{RL}\&(v)^U_{RL} =_{\mathbf{2}} \sum_{i,j\in_{\mathbf{2}} J} \check{a}_i\check{b}_j \cdot (\widehat{q}_i)^U_{RL}\&(\widehat{q}_j)^U_{RL} =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}} J} \check{a}_j\check{b}_j \cdot (\widehat{q}_j)^U_{RL} =_{\mathbf{2}} (u\cdot v)^U_{RL}]\!]_{\chi^U_{RL}(B_U)} = 1 \tag{5.5}
$$

Proof. By Theorem 5.38 and Corollary 5.30. □

Definition 5.41. Let $U \in \mathcal{U}$ and $q_j = \sigma_U(p_j)$ for each $j \in J$. Define $(\widehat{q}_i)_L\&(\widehat{q}_j)_R$ by

$$
\begin{cases}
\mathcal{D}\left((\widehat{q}_i)_L\&(\widehat{q}_j)_R\right) = \{\check{r} \mid r \in \mathbb{Q}\} \\
\left((\widehat{q}_i)_L\&(\widehat{q}_j)_R\right)(\check{r}) = \begin{cases} 1 & 1 \le r \\ (\chi_L(q_i)\&\chi_R(q_j))^\perp & 0 \le r < 1 \\ 0 & r < 0. \end{cases}
\end{cases}
$$

$(\widehat{q}_i)_L\&(\widehat{q}_j)_R$ is a real number in $V^{\chi_{LR}(B_U)}$ $(= V^{\mathbf{2}})$. For $a_i, b_j \in \mathbb{C}$ $(i, j \in J)$, let

$$
(\sum_{i\in_{\mathbf{2}} J} \check{a}_i \cdot \widehat{q}_i)_L\&(\sum_{j\in_{\mathbf{2}} J} \check{b}_j \cdot \widehat{q}_j)_R \stackrel{\text{def}}{=} \sum_{i,j\in_{\mathbf{2}} J} \breve{\check{a}}_i\check{b}_j \cdot (\widehat{q}_i)_L\&(\widehat{q}_j)_R, \tag{5.6}
$$

Lemma 5.42. *Let $U \in \mathcal{U}$.*

1. *If $q \in B_U$, then $q \neq 0 \Longleftrightarrow \chi^U_{LR}(q) = 1$.*

2. $\begin{cases} [\![(\widehat{q}_i)_L \& (\widehat{q}_j)_R =_{\mathbf{2}} \check{1}]\!]_{\mathcal{Q}} = 1, & i = j \\ [\![(\widehat{q}_i)_L \& (\widehat{q}_j)_R =_{\mathbf{2}} \check{0}]\!]_{\mathcal{Q}} = 1, & i \neq j. \end{cases}$

3. $[\![(\sum_{i \in_{\mathbf{2}} J} \check{a}_i \cdot \widehat{q}_i)_L \& (\sum_{j \in_{\mathbf{2}} J} \check{b}_j \cdot \widehat{q}_j)_R =_{\mathbf{2}} \sum_{i,j \in_{\mathbf{2}} J} \check{\bar{a}}_i \check{b}_j]\!] = 1.$
Hence, $(u)_L \& (v)_R$ *is a check set.*

Proof. 1. $q = \bigvee_{j \in K}$ for some $K \subset J$, where $q_j = \sigma_U(p_j)$.

By Theorem 3.18, $\chi_L(q_i) \& \chi_R(q_j)) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$.

Hence, $\chi^U_{LR}(q) = \bigvee_{j \in K} (\chi_L(q_j) \& \chi_R(q_j)) = \begin{cases} 1 & K \neq \emptyset \\ 0 & K = \emptyset. \end{cases}$

$$\therefore \quad q \neq 0 \Longleftrightarrow K \neq \emptyset \Longleftrightarrow \chi^U_{LR}(q) = 1.$$

2. By the definition of $(\widehat{q}_i)_L \& (\widehat{q}_j)_R$,

$$((\widehat{q}_i)_L \& (\widehat{q}_j)_R)(\check{r}) = \begin{cases} \check{1}(\check{r}) & i = j \\ \check{0}(\check{r}) & i \neq j, \end{cases} \quad \text{for } r \in \mathbb{Q} .$$

$$\therefore \quad \begin{cases} [\![(\widehat{q}_i)_L \& (\widehat{q}_j)_R =_{\mathbf{2}} \check{1}]\!]_{\mathcal{Q}} = 1, & i = j \\ [\![(\widehat{q}_i)_L \& (\widehat{q}_j)_R =_{\mathbf{2}} \check{0}]\!]_{\mathcal{Q}} = 1, & i \neq j. \end{cases}$$

3. $[\![(u)_L \& (v)_R =_{\mathbf{2}} \sum_{i,j \in_{\mathbf{2}} J} \check{\bar{a}}_i \check{b}_j \cdot (\widehat{q}_i)_L \& (\widehat{q}_j)_R =_{\mathbf{2}} \sum_{j \in_{\mathbf{2}} J} \check{\bar{a}}_j \check{b}_j]\!]_{\mathcal{Q}} = 1.$

□

5.8 Representation of states and propositions in $V^{\mathcal{Q}}$

For each $U \in \mathcal{U}$, Boolean $(\bigvee, ^{\perp})$-sublattice B_U of $P(\mathcal{H})$, is embedded in $\mathcal{Q}$ as $\chi_R(B_U)$, $\chi_L(B_U)$ and also $\chi^U_{RL}(B_U)$. Hence, Boolean valued universe V^{B_U} is embedded in $V^{\mathcal{Q}}$ as $V^{\chi_R(B_U)}$, $V^{\chi_L(B_U)}$ and $V^{\chi^U_{RL}(B_U)}$.

Vector x in $\mathcal{H}$ is represented by a normal operator

$$x = \sum_{j \in J} (Ue_j, x) \cdot \sigma_U(p_j) \quad \text{for each } U \in \mathcal{U},$$

and projection $q \in P(\mathcal{H})$ is represented by real number $\widehat{q}$ in $V^{P(\mathcal{H})}$, where

$$q = \bigvee_{j \in K} \sigma_U(p_j) = \sum_{j \in K} \sigma_U(p_j) \quad \text{for some } U \in \mathcal{U} \text{ and } K \subset J.$$

As defined in §5.6,

$$
\begin{aligned}
\widehat{x}_U &\stackrel{\text{def}}{=} \sum_{j\in_2 J} (Ue_j,x)^{\check{}}\cdot\widehat{\sigma_U(p_j)} \in V^{B_U} \quad \text{for a vector } x \text{ in } \mathcal{H} \text{ and } U\in\mathcal{U},\\
\widehat{\mathcal{H}}_U &\stackrel{\text{def}}{=} \{<U,\widehat{x}_U>|\, x\in\mathcal{H}\} \quad \text{for } U\in\mathcal{U},\\
\widehat{x} &\stackrel{\text{def}}{=} \{<U,\widehat{x}_U>|\, U\in\mathcal{U}\} \quad \text{for a vector } x \text{ in } \mathcal{H},\\
\widehat{\mathcal{H}} &\stackrel{\text{def}}{=} \bigcup_{U\in\mathcal{U}} \widehat{\mathcal{H}}_U.
\end{aligned}
$$

$\widehat{x}_U$ is a complex number in V^{B_U}. Thus, x is represented in $V^{\mathcal{Q}}$ as right-complex number and left-complex number for each $U\in\mathcal{U}$.

$$
\begin{aligned}
(\widehat{x}_U)_R &= \sum_{j\in_2 J} (Ue_j,x)^{\check{}}\cdot(\widehat{\sigma_U(p_j)})_R \in V^{\chi_R(B_U)},\\
(\widehat{x}_U)_L &= \sum_{j\in_2 J} (Ue_j,x)^{\check{}}\cdot(\widehat{\sigma_U(p_j)})_L \in V^{\chi_L(B_U)}.
\end{aligned}
$$

The coordinates of $(\widehat{x}_U)_R$ in $V^{\chi_R(B_U)}$ and $(\widehat{x}_U)_L$ in $V^{\chi_L(B_U)}$ are check sets for each $U\in\mathcal{U}$. $(\widehat{x}_U)_R$ is represented by coordinate matrix denoted by $|x\rangle_U \in V$; $(\widehat{x}_U)_L$ is represented by coordinate matrix denoted by $\langle x|_U \in V$, using the Dirac's notation:

$$
|x\rangle_U = \begin{bmatrix} \vdots \\ (Ue_j,x) \\ \vdots \end{bmatrix}, \quad \langle x|_U = (|x\rangle_U)^* = \begin{bmatrix} \cdots & \overline{(Ue_j,x)} & \cdots \end{bmatrix},
$$

Let

$$
\begin{aligned}
&\mathrm{Sh}_U(\mathcal{H}_R) =< \widetilde{\mathcal{H}}_R, \pi_R >, \quad \mathrm{Sh}_U(\mathcal{H}_L) =< \widetilde{\mathcal{H}}_L, \pi_L >, \quad \text{where}\\
&\widetilde{\mathcal{H}}_R \stackrel{\text{def}}{=} \{<U,|x\rangle_U>|\, U\in\mathcal{U},\ x\in\mathcal{H}\}, \quad \pi_R :< U,|x\rangle_U > \mapsto U,\\
&\widetilde{\mathcal{H}}_L \stackrel{\text{def}}{=} \{U,\langle x|_U \mid U\in\mathcal{U},\ x\in\mathcal{H}\}, \quad \pi_L :< U,\langle x|_U > \mapsto U,
\end{aligned}
$$

$x\in\mathcal{H}$ is represented by a cross-section $|x\rangle$ of the sheaf $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}_R)$ of Hilbert space over $\mathcal{U}$, and also represented by a cross-section $\langle x|$ of the sheaf $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}_L)$ of Hilbert space over $\mathcal{U}$.

$$
|x\rangle = \{<U,|x\rangle_U>|\, U\in\mathcal{U}\}, \quad \langle x| = \{<U,\langle x|_U>|\, U\in\mathcal{U}\}
$$

Theorem 5.43. *If $U\in\mathcal{U}$ is represented by matrix $[U]$, i.e.*

$$
[U] = \begin{bmatrix} & \vdots & \\ \cdots & (Ue_i,e_j) & \cdots \\ & \vdots & \end{bmatrix} = [(Ue_i,e_j)]_{ij}.
$$

then

$$|x\rangle_U = [U]\cdot|x\rangle_E,$$

where E is the identity in $\mathcal{U}$.

Proof.

$$x = \sum_{j\in J}(e_j,x)e_j = \sum_{j\in J}(e_j,x)\sum_{i\in J}(Ue_i,e_j)Ue_i$$
$$= \sum_{i\in J}(\sum_{j\in J}(Ue_i,e_j)(e_j,x))Ue_i.$$

Hence,

$$|x\rangle_U = \begin{bmatrix} \vdots \\ (Ue_i,x) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ (\sum_{j\in J}((e_j,x)(Ue_i,e_j))) \\ \vdots \end{bmatrix} = U\cdot|x\rangle_E.$$

□

By Theorem 5.40 we have:

$$[\![(\widehat{x}_U)^U_{RL}\&(\widehat{y}_U)_R =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}}J}(Ue_j,x)\check{}\cdot(Ue_j,y)\check{}\cdot(\widehat{q}_j)_R]\!]_{\chi_R(B_U)^U} = 1, \tag{5.7}$$

$$[\![(\widehat{x}_U)_L\&(\widehat{y}_U)^U_{RL} =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}}J}\overline{(Ue_j,x)}\check{}\cdot(Ue_j,y)\check{}\cdot(\widehat{q}_j)_L]\!]_{\chi_L(B_U)} = 1, \tag{5.8}$$

$$[\![(\widehat{x}_U)^U_{RL}\&(\widehat{y}_U)^U_{RL} =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}}J}(Ue_j,x)\check{}\cdot(Ue_j,y)\check{}\cdot(\widehat{q}_j)^U_{RL}]\!]_{\chi^U_{RL}(B_U)} = 1. \tag{5.9}$$

By Lemma 5.42, if $x,y\in\mathcal{H}$, then $(\widehat{x}_U)_L\&(\widehat{y}_U)_R$ is a check set representing the inner product:

$$[\![(\widehat{x}_U)_L\&(\widehat{y}_U)_R =_{\mathbf{2}} \sum_{j\in_{\mathbf{2}}J}\overline{(Ue_j,x)}\check{}\cdot(Ue_j,y)\check{}]\!]_{\mathcal{Q}} = 1. \tag{5.10}$$

Propositions are represented by projections, and each projection p are represented by a real number $\widehat{p}$ in some V^{B_U}.

Theorem 5.44. *If p is a projection, then there exists $U\in\mathcal{U}$ such that $\widehat{p}$ is a real number in $V^{B_U}\subset V^{P(H)}$. Then for vector x in $\mathcal{H}$,*

$$(\widehat{p})_{RL}\&(\widehat{x}_U)_R = ((\widehat{px})_U)_R.$$

Proof. Let $q_j = \sigma_U(p_j)$.

$$\widehat{p} = \sum_{j\in_{\mathbf{2}}J}\check{a}_j\cdot\widehat{q}_j,$$

where $a_j = 1$ or 0 for each $j\in J$. Hence, by Theorem 5.40. □

By Theorem 5.44, '$(\widehat{p})^U_{RL}\&-$' induces the mapping $|x\rangle\to|p(x)\rangle$, and is denoted by $|p\rangle\langle p|$.

$$|p\rangle\langle p| : |x\rangle\mapsto|p(x)\rangle.$$

6 An interpretation of quantum theory in $V^{\mathcal{Q}}$

6.1 Rules in quantum theory

A general mathematical framework within which quantum theory is developed is described as the following four rules.

Rule 1. The physical notion of 'state' of the system is represented as a vector in a complex Hilbert space $\mathcal{H}$.

Rule 2. Observables are represented by self-adjoint operators which act on the Hilbert space $\mathcal{H}$.

Rule 3. If an observable quantity $\mathcal{A}$ is represented by self-adjoint operator A, and a state is represented by a normalized vector $\psi \in \mathcal{H}$, then the expected result $< A >_\psi$ of measuring $\mathcal{A}$ is

$$< A >_\psi = < \psi, A\psi > .$$

Rule 4. In the absence of any external influence (i.e. in a closed system), the state vector ψ_t changes smoothly in time t according to the time-dependent Schrödinger equation

$$i\hbar \frac{d\psi_t}{dt} = \mathrm{H}\psi_t,$$

where H is a special operator known as the Hamiltonian.

6.2 State space $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ in $V^{\mathcal{Q}}$

A state of the physical system which is represented by a vector in $\mathcal{H}$ is represented in $V^{P(\mathcal{H})}$ as a complex number. In Section 5.8, we saw that each vector x in $\mathcal{H}$ is represented by cross-section $|x\rangle$ of $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}_R)$ and $\langle x|$ of $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H}_L)$:

$$|x\rangle = \{< U, |x\rangle_U >| \, U \in \mathcal{U}\} = \{< U, [U] \cdot |x\rangle_E >| \, U \in \mathcal{U}\},$$

$$\langle x| = \{< U, \langle x|_U >| \, U \in \mathcal{U}\} = \{< U, [U] \cdot \langle x|_E >| \, U \in \mathcal{U}\},$$

where E is the unit element of $\mathcal{U}$.

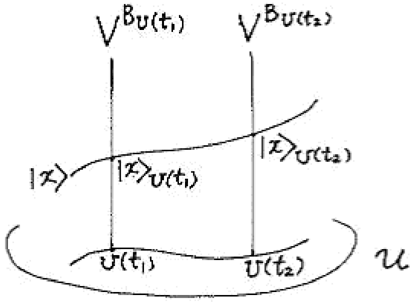
For unitary operator $U \in \mathcal{U}$ there exists a self-adjoint operator A such that $U = e^{iA}$. Let $U(t) \in \mathcal{U}$ by

$$U(t) = e^{iAt} \quad (t \in \mathbb{R}).$$

$\{e^{iAt}\}_{t\in\mathbb{R}}$ forms a strongly continuous one parameter group in $\mathcal{U}$. The parameter t may be considered as 'time'. Let $A = -\frac{\mathrm{H}}{\hbar}$.

$$\begin{aligned} |x\rangle_{U(t)} \quad &= \quad [U(t)] \cdot |x\rangle_E = [e^{-i\frac{\mathrm{H}}{\hbar}t}] \cdot |x\rangle_E \\ \therefore \quad & \frac{d}{dt}|x\rangle_{U(t)} = \frac{d}{dt}[e^{-i\frac{\mathrm{H}}{\hbar}t}] \cdot |x\rangle_E \\ \therefore \quad & i\hbar\frac{d}{dt}|x\rangle_{U(t)} = \mathrm{H}\,|x\rangle_{U(t)}. \end{aligned}$$

Thus, $|x\rangle_{U(t)}$ is considered as a vector-valued function of t which is the Schrödinger's equation.



6.3 Propositions $\{(\widehat{q})^U_{RL} \mid q \in B_U,\ U \in \mathcal{U}\}$ in $V^{\mathcal{Q}}$

An observable which is represented by a self-adjoint operator on $\mathcal{H}$ is represented in $V^{P(\mathcal{H})}$ as a real number. A proposition is an observable and represented in $V^{P(\mathcal{H})}$ as a real number u such that $[\![u =_{\mathbf{T}} \check{1} \leftrightarrow_{\mathbf{2}} (u =_{\mathbf{T}} \check{0})^{\perp}]\!]_{P(\mathcal{H})} = 1$. Namely,

$$[\![u =_{\mathbf{T}} \check{1} \vee u =_{\mathbf{T}} \check{0}]\!]_{P(\mathcal{H})} = 1.$$

u is the real number which represents projections in $P(\mathcal{H})$.

Let q be a projection. There exists a unitary operator $U \in \mathcal{U}$ such that $q \in B_U$ by Theorem 3.9, and q is represented by an operator $|q\rangle\langle q|$ (cf. Theorem 5.44).

Let $x \in \mathcal{H}$ be a unit vector representing a state. Product of $\widehat{q}$ and $\widehat{x}_U$ is defined, and by Corollary 5.30

$$\widehat{q} \cdot \widehat{x}_U = (\sum_{j \in J} \check{a}_j \cdot \widehat{\sigma_U(p_j)}) \cdot (\sum_{j \in J} (Ue_j, x) \cdot \widehat{\sigma_U(p_j)}) = (\widehat{q(x)})_U.$$

The state after the observation is represented by cross-section

$$|q(x)\rangle = \{< U',\, [U'U^*] \cdot |q(x)\rangle_U > \mid\ U' \in \mathcal{U}\}.$$

Let $\||q(x)\rangle\| = \sqrt{\langle q(x)|\&|q\rangle}$. Then $\||q(x)\rangle\| = \|q(x)\|$.

- The probability that the proposition is 'yes' in the state represented by x is $\||q(x)\rangle\|^2$.
- Normalized state after the observation is represented by $\frac{|q(x)\rangle}{\||q(x)\rangle\|}$.

Operator $|q\rangle\langle q| : |x\rangle \mapsto |qx\rangle$ on $\mathrm{Sh}_{\mathcal{U}}(\mathcal{H})$ is determined by

$$\widehat{x_U} \mapsto \widehat{q} \cdot \widehat{x_U} = \widehat{(q(x))}_U \quad \text{in } V^{B_U}\ .$$

That is,

$$|q\rangle\langle q| : \mathrm{Sh}_{\mathcal{U}}\mathcal{H} \to \mathrm{Sh}_{\mathcal{U}}\mathcal{H} \quad \text{defined by } |q\rangle\langle q|(|x\rangle) = |q(x)\rangle.$$

Let P and Q be propositions represented by $p \in B_{U_1}$ and $q \in B_{U_2}$, respectively. And assume that P is observed 'and then' Q is observed in a state $x \in \mathcal{H}$. First, P is observed in the universe $V^{B_{U_1}}$. The resulting state $|p\rangle\langle p|(|x\rangle) = |p(x)\rangle$ is determined by $\widehat{p} \cdot \widehat{x}_{U_1} = (\widehat{p(x)})_{U_1}$, i.e.

$$|p(x)\rangle = \{< U,\, [UU_1^*] \cdot |p(x)\rangle_{U_1} >|\ U \in \mathcal{U}\}.$$

Then Q is observed in $V^{B_{U_2}}$. In order to observe Q in the state $|p(x)\rangle$, we have to move to the universe $V^{B_{U_2}}$, where the state $|p(x)\rangle$ is

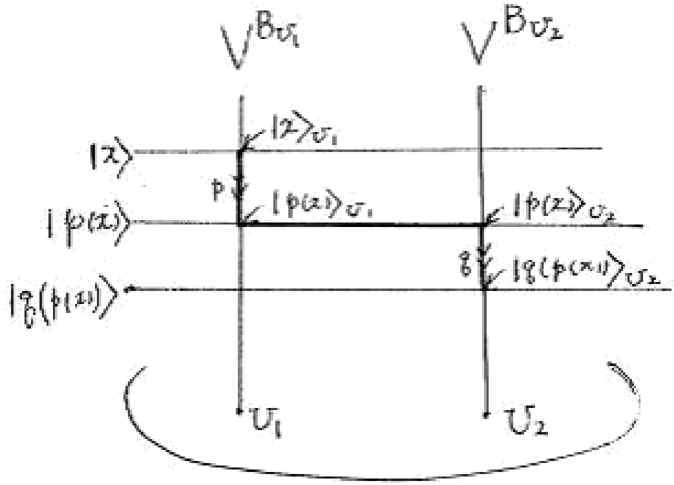
$$|p(x)\rangle_{U_2} = [U_2U_1^*] \cdot |p(x)\rangle_{U_1}.$$

$|q\rangle\langle q|(|p(x)\rangle)$ is determined by $\widehat{q} \cdot \widehat{p(x)}_{U_2} = \widehat{q(p(x))}_{U_2}$.

$$|q\rangle\langle q|(|p(x)\rangle) = \{< U,\, [UU_2^*] \cdot |q(p(x))\rangle_{U_2} >|\ U \in \mathcal{U}\}.$$

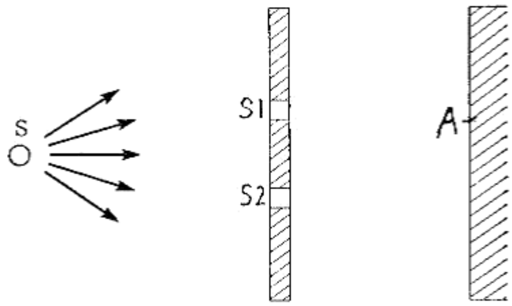
That is, 'P and then Q', in symbols $P\&Q$, is represented by the composition of morphisms $|p\rangle\langle p|$ and $|q\rangle\langle q|$.

$$|q\rangle\langle q|(|p\rangle\langle p|(|x\rangle)) = |q(p(x))\rangle.$$



6.4 Application 1 : Double slit experiment

The double-slit experiment is taken as evidence of the "wave-particle duality" predicted by quantum physics: In the following figure, S is a source of light. The light emitted from S passes through two slits $S1$ and $S2$ on a screen, and then is viewed in a rear screen.



When either slit is covered, a single peak of wave energy is observed on the screen from the light passing through the other slit. But when both slits are open, instead of the sum of these two singular peaks that would be expected if light were made of particles, a pattern of light and dark fringes is observed.

Let x be a unit vector in a Hilbert space $\mathcal{H}$ which represents the state of a photon emitted from S, and

$P^{(A)}$ be proposition "the photon comes to A on the rear screen",

$P^{(i)}$ ($i = 1, 2$) be proposition "the photon passes through slit Si ".

$p^{(i)}$ ($i = 1, 2$) be the projection representing $P^{(i)}$.

Then $p^{(1)} = (p^{(2)})^{\perp}$, hence $p^{(1)} \rfloor_{\circ}\, p^{(2)}$ and $(p^{(1)} \vee p^{(2)}) \rfloor_{\circ}\, p^{(A)}$. It follows that

$$(P^{(1)} \vee P^{(2)}) \wedge P^{(A)} = (P^{(1)} \vee P^{(2)}) \& P^{(A)} = (P^{(1)} \& P^{(A)}) \vee (P^{(2)}) \& P^{(A)}).^{2}$$

Note that each $p^{(i)}$ $(i = 1, 2)$ is not compatible with $p^{(A)}$ generally. The result of measurement of proposition $(P^{(1)} \vee P^{(2)}) \wedge P^{(A)}$ in state x is represented by $|p^{(A)}((p^{(1)} + p^{(2)})(x))\rangle$, that is $|p^{(A)}((p^{(1)}(x)) + p^{(A)}(p^{(2)}(x))\rangle$ with expectation value

$$||p^{(A)}((p^{(1)}(x)) + p^{(A)}(p^{(2)}(x))||^{2}.$$

If slit S2 is covered, then $p^{(1)} \rfloor_{\circ}\, p^{(A)}$. That is, $p^{(1)}$ and $p^{(A)}$ are measured in a Boolean valued universe which is a stalk of $\mathrm{Sh}_{\mathcal{U}}(V^{B})$. Similarly, if slit S1 is covered, then $p^{(2)} \rfloor_{\circ}\, p^{(A)}$. That is, $p^{(2)}$ and $p^{(A)}$ are measured in another Boolean valued universe. Hence,

$$P^{(1)} \& P^{(A)} = P^{(1)} \wedge P^{(A)} \quad \text{and} \quad P^{(2)} \& P^{(A)} = P^{(2)} \wedge P^{(A)}.$$

[2] If P, Q are propositions represented by projections p, q, then 'P and then Q', in symbols $P \& Q$, is represented by the composition $p \& q$ of projections p, q:

$$(p \& q)(x) = q(p(x)) \quad \text{for } x \in \mathcal{H}.$$

& is distributive over arbitrary supremum, and if p, q are compatible then $p \& q = p \wedge q$.

The result of measurement of proposition "$P^{(i)}$ and then $P^{(A)}$" in state x is represented by $|p^{(A)}(p^{(i)}(x))\rangle$ with expectation value for $\|p^{(A)}(p^{(i)}(x))\|^2$ for $i = 1, 2$. So one would expect, even when both slits are open, that

$$(P^{(1)} \vee P^{(2)}) \wedge P^{(A)} = (P^{(1)} \wedge P^{(A)}) \vee (P^{(2)}) \wedge P^{(A)}). \tag{6.1}$$

If (6.1) were true, then $(p^{(1)} \wedge p^{(A)}) \perp (p^{(2)} \wedge p^{(A)})$ and the expectation value of $(P^{(1)} \vee P^{(2)}) \wedge P^{(A)}$ in state x is

$$\|p^{(A)}(p^{(1)}(x))\|^2 + \|p^{(A)}(p^{(2)}(x))\|^2,$$

instead of

$$\|p^{(A)}((p^{(1)}(x)) + p^{(A)}(p^{(2)}(x))\|^2.$$

However, if both slits are open, $p^{(A)}$ is not compatible with each of $p^{(1)}$ and $p^{(2)}$, hence,

$$P^{(i)} \& P^{(A)} \neq P^{(i)} \wedge P^{(A)} \text{ for } i = 1, 2.$$

$P^{(1)} \& P^{(A)}$ and $P^{(2)} \& P^{(A)}$ are measured in different universes.

Recall that the result of measurement of proposition P represented by $p \in B_{U_1}$ in a state $x \in \mathcal{H}$ is given by $|p(x)\rangle_{U_1}$, which determines $|p(x)\rangle$:

$$|p(x)\rangle = \{< U, |p(x)\rangle_U >| \, U \in \mathcal{U}\} \text{ and } |p(x)\rangle_U = [U] \cdot [U_1^*] \cdot |p(x)\rangle_{U_1}.$$

Let $p^{(1)}$ and $p^{(2)}$ are in B_{U_1} and $p^{(A)}$ is in B_{U_2} where $U_1, U_2 \in \mathcal{U}$. Then the result of measurement of $P^{(i)}$ in state $x \in \mathcal{H}$ is given by $|p^{(i)}(x)\rangle$ which is determined by $|p^{(i)}(x)\rangle_{U_1}$ in the universe $V^{B_{U_1}}$, i.e.

$$|p^{(i)}(x)\rangle = \{< U, [U][U_1]^* |p^{(i)}(x)\rangle_{U_1} >| \, U \in \mathcal{U}\}.$$

The result of measurement of $(P^{(1)} \vee P^{(2)}) \& P^{(A)}$ in a state $x \in \mathcal{H}$ is given by

$$|p^{(A)}\rangle\langle p^{(A)}|(|p^{(1)}(x)\rangle + |p^{(2)}(x)\rangle) = |p^{(A)}(p^{(1)}(x) + p^{(2)}(x))\rangle,$$

which is determined by $|p^{(A)}((p^{(1)} + p^{(2)})(x))\rangle_{U_2}$ in the universe $V^{B_{U_2}}$.

$$|p^{(A)}((p^{(1)} + p^{(2)})(x)\rangle_{U_2} = |p^{(A)}(p^{(1)}(x))\rangle_{U_2} + |p^{(A)}(p^{(2)}(x))\rangle_{U_2},$$

where $|p^{(A)}(p^{(1)}(x))\rangle_{U_2}$ and $|p^{(A)}(p^{(2)}(x))\rangle_{U_2}$ are not orthogonal. Hence,

$$\|p^{(A)}((p^{(1)} + p^{(2)})(x))\|^2 \neq \|p^{(A)}(p^{(1)}(x))\|^2 + \|p^{(A)}(p^{(2)}(x))\|^2$$

That is,

$$(P^{(1)} \vee P^{(2)}) \wedge P^{(A)} \neq (P^{(1)} \wedge P^{(A)}) \vee (P^{(2)}) \wedge P^{(A)}).$$

The difference comes from the fact that $p^{(1)}$, $p^{(2)}$ and $p^{(A)}$ are not in one Boolean universe.

6.5 Application 2 : Entangled states of spin and Bell's inequality

We consider a particle whose decay produces two spin-$\frac{1}{2}$ particles, 1 and 2, whose total spin angular momentum is zero. These particles move away from each other in opposite directions, and the components of these spins along various directions are subsequently measured by two observers, A and B, say. The constraint on the total spin means that if observers agree to measure the spin along a particular direction $\mathbf{n}$ and if A measures $+\frac{1}{2}\hbar$, then B will measures $-\frac{1}{2}\hbar$, and if A measures $-\frac{1}{2}\hbar$, then B will measures $+\frac{1}{2}\hbar$.

If one can concentrates purely on its internal-spin properties, a state of an particle can be described by unit vector in $\mathcal{H} = \mathbb{C}^2$. In this system, observables are represented by self-adjoint operator that act on the states.

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be unit vectors representing the eigen-vectors in which a particle has spin $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, respectively, along z-axis. $\{|\uparrow\rangle, |\downarrow\rangle\}$ is a basis of $\mathbb{C}^2$, with respect to which the observable is represented by

$$\frac{1}{2}\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The state of pair of spin-$\frac{1}{2}$ particles is described a unit vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$. The state of pair of spin-$\frac{1}{2}$ particles whose total spin angular momentum is zero can be written in terms of the associated eigen-vectors as

$$\psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),$$

where $|\uparrow\rangle|\downarrow\rangle$ $(=|\uparrow\rangle \otimes |\downarrow\rangle)$ is the state in which particles 1 and 2 have spin $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, respectively, along z axis, and vice versa for $|\downarrow\rangle|\uparrow\rangle$.

By the constraint on the total spin, the state ψ of pair is determined by the state of particle 1. In what follows we write for simplicity as

$$\psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).$$

Bell's inequality

Now let us consider Bell's inequality, which is violated in quantum theory, following the description in J.J.Sakurai [8].

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be unit vectors which are, in general, not mutually orthogonal. For $\mathbf{n} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ let us consider propositions $P(i, \mathbf{n}, +)$ and $P(i, \mathbf{n}, -)$ for $i = 1, 2$.

$P(i, \mathbf{n}, +)$: "observer measures the spin of particle i to be $+$ along direction $\mathbf{n}$"

$P(i, \mathbf{n}, -)$: "observer measures the spin of particle i to be $-$ along direction $\mathbf{n}$"

Then obviously we have $P(i,\mathbf{n},-) = P(i,\mathbf{n},+)^{\perp}$ for $i = 1, 2$. Hence, the following proposition holds for all $\mathbf{m},\mathbf{n} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}$:

$$\Big(P(1,\mathbf{m},-) \vee P(1,\mathbf{m},+)\Big) \wedge \Big(P(2,\mathbf{n},-) \vee P(2,\mathbf{n},+)\Big).$$

By the classical logic, proposition $\forall \mathbf{m} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}(P(1,\mathbf{m},-) \vee P(1,\mathbf{m},+))$ is equivalent to the following proposition by the logical distributive law:

$$[P(1,\mathbf{a},+) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},+)] \vee [P(1,\mathbf{a},+) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},-)] \vee$$

$$[P(1,\mathbf{a},+) \wedge P(1,\mathbf{b},-) \wedge P(1,\mathbf{c},+)] \vee [P(1,\mathbf{a},+) \wedge P(1,\mathbf{b},-) \wedge P(1,\mathbf{c},-)] \vee$$

$$[P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},+)] \vee [P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},-)] \vee$$

$$[P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},-) \wedge P(1,\mathbf{c},+)] \vee [P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},-) \wedge P(1,\mathbf{c},-)].$$

'$P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},+)$' , for example, means :

> 'If the particle 1 is measured along the direction **a**, we obtain a minus sign; if it is measured along the direction **b**, we obtain a plus sign; if it is measured along the direction **c**, we obtain a plus sign'.

If '$[P(1,\mathbf{a},-) \wedge P(1,\mathbf{b},+) \wedge P(1,\mathbf{c},+)]$', we say 'particle 1 belongs $(\mathbf{a}_-,\mathbf{b}_+,\mathbf{c}_+)$'. Then there must be a perfect matching in the sense that the other particle necessarily belongs to type $(\mathbf{a}+,\mathbf{b}-,\mathbf{c}-)$ to ensure zero total angular momentum. In any given event, the particle pair in question must be a member of one of the eight types shown in the following table.

Population	Particle 1	Particle 2
N_1	$(\mathbf{a}_+,\mathbf{b}_+,\mathbf{c}_+)$	$(\mathbf{a}_-,\mathbf{b}_-,\mathbf{c}_-)$
N_2	$(\mathbf{a}_+,\mathbf{b}_+,\mathbf{c}_-)$	$(\mathbf{a}_-,\mathbf{b}_-,\mathbf{c}_+)$
N_3	$(\mathbf{a}_+,\mathbf{b}_-,\mathbf{c}_+)$	$(\mathbf{a}_-,\mathbf{b}_+,\mathbf{c}_-)$
N_4	$(\mathbf{a}_+,\mathbf{b}_-,\mathbf{c}_-)$	$(\mathbf{a}_-,\mathbf{b}_+,\mathbf{c}_+)$
N_5	$(\mathbf{a}_-,\mathbf{b}_+,\mathbf{c}_+)$	$(\mathbf{a}_+,\mathbf{b}_-,\mathbf{c}_-)$
N_6	$(\mathbf{a}_-,\mathbf{b}_+,\mathbf{c}_-)$	$(\mathbf{a}_+,\mathbf{b}_-,\mathbf{c}_+)$
N_7	$(\mathbf{a}_-,\mathbf{b}_-,\mathbf{c}_+)$	$(\mathbf{a}_+,\mathbf{b}_+,\mathbf{c}_-)$
N_8	$(\mathbf{a}_-,\mathbf{b}_-,\mathbf{c}_-)$	$(\mathbf{a}_+,\mathbf{b}_+,\mathbf{c}_+)$

These eight possibilities are mutually exclusive and disjoint. The population of each type is indicated in the first column.

Let us consider '$P(1,\mathbf{a},+) \wedge P(2,\mathbf{b},+)$' . It is clear from the table that the pair belong to either type 3 or type 4, so the number of particle pairs for which the situation is realized is $N_3 + N_4$. Because $N_i \geq 0$ $(i = 1 \cdots 8)$, we have inequality relation like

$$N_3 + N_4 \leq (N_2 + N_4) + (N_3 + N_7).$$

Let $p(\mathbf{a}+;\mathbf{b}+)$ be the probability that, in the random selection, '$[P(1,\mathbf{a},+) \wedge P(2,\mathbf{b},+)$' holds, and so on. That is, $p(\mathbf{a}\pm;\mathbf{b}\pm)$ is the probability of $P(\mathbf{a}\pm;\mathbf{b}\pm)$, where

$$\begin{aligned}
P(\mathbf{a}+;\mathbf{b}+) &= P(1,\mathbf{a},+) \wedge P(2,\mathbf{b},+) \\
P(\mathbf{a}+;\mathbf{b}-) &= P(1,\mathbf{a},+) \wedge P(2,\mathbf{b},-) \\
P(\mathbf{a}-;\mathbf{b}+) &= P(1,\mathbf{a},-) \wedge P(2,\mathbf{b},+) \\
P(\mathbf{a}-;\mathbf{b}-) &= P(1,\mathbf{a},-) \wedge P(2,\mathbf{b},-)
\end{aligned}$$

Clearly, we have

$$p(\mathbf{a}+;\mathbf{b}+) = \frac{(N_3 + N_4)}{\sum_{i=1}^{8} N_i}.$$

In a similar manner, we obtain

$$p(\mathbf{a}+;\mathbf{c}+) = \frac{(N_2 + N_4)}{\sum_{i=1}^{8} N_i} \quad \text{and} \quad p(\mathbf{c}+;\mathbf{b}+) = \frac{(N_3 + N_7)}{\sum_{i=1}^{8} N_i}.$$

The possibility condition now becomes

$$p(\mathbf{a}+;\mathbf{b}+) \leq p(\mathbf{a}+;\mathbf{c}+) + p(\mathbf{c}+;\mathbf{b}+).$$

This is the Bell's inequality, which is not compatible with quantum mechanical prediction. The reason is that the logical distributive law was applied to incompatible set $\{P(i,\mathbf{n},\pm) \mid i \in \{1,2\},\ \mathbf{n} \in \{\mathbf{a},\mathbf{b},\mathbf{c}\}\}$.

Bell's inequality in the universe $V^{Q(P(\mathbb{C}^2))}$

Now let us evaluate $p(\mathbf{a}+;\mathbf{b}+)$ in the universe $V^{Q(P(\mathbb{C}^2))}$. In this section we use the same notation to denote projection as the proposition represented by the projection. As seen in sections 5.8 and 5.9, states are considered as cross-sections of sheaf $\mathrm{Sh}_{\mathcal{U}}(\mathbb{C}^2)$, and propositions are considered as operations on the sheaf, i.e. proposition $P(\mathbf{a}+;\mathbf{b}+)$ is represented by

$$|P(\mathbf{a}+;\mathbf{b}+)\rangle\langle P(\mathbf{a}+;\mathbf{b}+)| : \mathrm{Sh}\,(\mathbb{C}^2) \to \mathrm{Sh}\,(\mathbb{C}^2).$$

Let $\alpha = \mathbf{a},\mathbf{b},\mathbf{c}$, and let $|\alpha+\rangle$ and $|\alpha-\rangle$ be vectors representing the eigen-states of proposition $P(1,\alpha,+)$ with eigen-value $+1$ and -1, respectively. $\{|\alpha+\rangle, |\alpha-\rangle\} \subset P(\mathbb{C}^2)$ forms a complete orthogonal system. Let $U_\alpha(e_1) = |\alpha+\rangle$, $U_\alpha(e_2) = |\alpha-\rangle$ and

B_{U_α} be the Boolean $(\bigvee, {}^\perp)$-sublattice of $P(\mathbb{C}^2)$ generated by $\{|\alpha+\rangle, |\alpha-\rangle\}$.

Coordinates of $|\mathbf{a}+\rangle$ and $|\mathbf{a}-\rangle$ with respect to basis $\{|\mathbf{a}+\rangle, |\mathbf{a}-\rangle\}$ are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$, respectively. That is,

$$|\mathbf{a}+\rangle_{U_a} = \begin{bmatrix}1\\0\end{bmatrix} \qquad |\mathbf{a}-\rangle_{U_a} = \begin{bmatrix}0\\1\end{bmatrix} \qquad |\psi\rangle_{U_a} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

The state after the observation of proposition $P(1, \mathbf{a}, +)$ in the state ψ is

$$|P(1, \mathbf{a}, +)\psi\rangle_{U_a} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} |\psi\rangle_{U_a} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, we have to change the universe from $V^{B_{U_a}}$ to $V^{B_{U_b}}$, for the observation of proposition $P(1, \mathbf{b}, -)$. $\mathbf{b}$ is a new axis that make an angle θ_{ab} with $\mathbf{a}$. We denote $\frac{\theta_{ab}}{2}$ by θ.

Coordinates of eigen-states $|\mathbf{b}+\rangle$ and $|\mathbf{b}-\rangle$ of proposition $P(1, \mathbf{b}, -)$ with eigen-value $+1$ and -1 with respect to P_{U_a} are $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, respectively. That is, the coordinate transformation $U_{b,a}$ is

$$[U_{b,a}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

By Theorem 5.43, $|P(1, \mathbf{a}, +)\psi\rangle_{U_b} = [U_{b,a}] \cdot |P(1, \mathbf{a}, +)\psi\rangle_{U_a}$.

Since $P(2, \mathbf{b}, +)$ is same as $P(1, \mathbf{b}, -)$, we denote them by $P(\mathbf{b}, -)$. The result of measuring

"$P(1, \mathbf{a}, +)$ and then $P(2, \mathbf{b}, +)$"

in state $|\psi\rangle$ is represented by

$$|P(\mathbf{b}, -)((P(\mathbf{a}, +)(\psi)))\rangle, \quad \text{where}$$

$$|P(\mathbf{a}, +)(\psi)\rangle_{U_a} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|P(\mathbf{a}, +)(\psi)\rangle_{U_b} = [U_{b,a}] \cdot \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

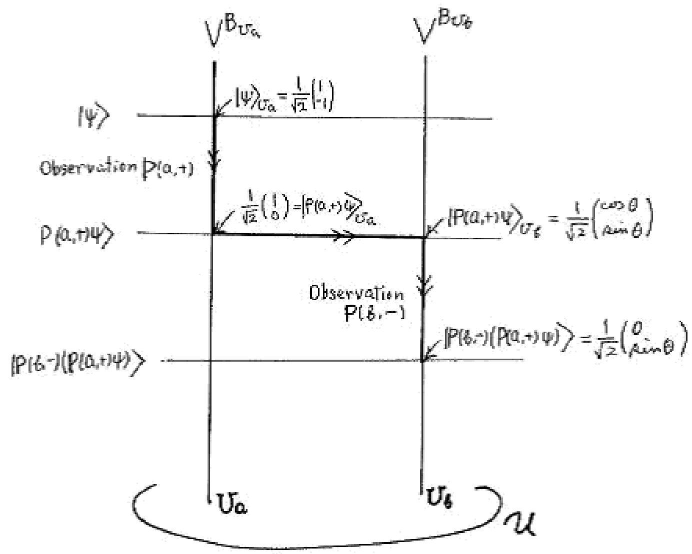
Hence,

$$|\, P(\mathbf{b}, -)((P(\mathbf{a}, +)(\psi)))\,\rangle_{U_b} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{2}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix}$$

It follows that

$$p(\mathbf{a}+; \mathbf{b}+) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sin\theta \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix} = \frac{1}{2} \sin^2\theta = \frac{1}{2} \sin^2 \left(\frac{\theta_{ab}}{2} \right).$$

Similarly, $p(\mathbf{a}+; \mathbf{c}+) = \frac{1}{2} \sin^2(\frac{\theta_{ac}}{2})$ and $p(\mathbf{c}+; \mathbf{b}+) = \frac{1}{2} \sin^2(\frac{\theta_{cb}}{2})$.



Now we assume that **a** points along the z-axis and that **b**, **c** lies in the x-z plain. If $\theta_{ac} = \theta_{cb} = \frac{\pi}{4}$, then

$$p(\mathbf{a}+;\mathbf{b}+) = \frac{1}{2}\sin^2(\frac{\pi}{4}) = \frac{0.5}{2}, \quad p(\mathbf{a}+;\mathbf{c}+) = p(\mathbf{c}+;\mathbf{b}+) = \frac{1}{2}\sin^2(\frac{\pi}{8}) \fallingdotseq \frac{0.14}{2}.$$

Hence, we obtain the result of quantum mechanics which is inconsistent with Bell's inequality.

$$p(\mathbf{a}+;\mathbf{b}+) \not\leq p(\mathbf{a}+;\mathbf{c}+) + p(\mathbf{c}+;\mathbf{b}+).$$

References

[1] G.Birkhoff and J.von Neumann, The logic of Quantum Mechanics, Ann. Math., 37, 1936, 823-843

[2] P.A.M.Dirac, The Principles of Quantum Mechanics, 4th edition, Oxford University Press, 1958.

[3] C.J.Mulvey and J.Wick Pelletier, Quantization of the Calculus of Relations, Canadian Mathematical Society, Conference Proceeding, Vol. 13, 1992.

[4] J.von Neumann, Mathematical foundation of quantum mechanics, Princeton University Press, 1955

[5] M. Ozawa, Transfer principle in quantum set theory, J. Symbolic Logic 72, 2007, 625-648

[6] C.Piron, Foundations of Quantum Physics, W.A. Benjamin, Inc. 1976

[7] P.Resende, Quantales and observational semantics, Current Reseach in Operational Quantum Logic, eds. B.Coecke, D.Moore and A.Wilce, Kluwer Academic Publishers (2000) pp. 263-288

[8] J.J.Sakurai, Modern Quantum Mechanics Addison-Wesley Pub.Co.,1994

[9] Michio Takano, Strong Completeness of Lattice Valued Logic, Archive for Mathematical Logic 41(2002) pp497-505

[10] G.Takeuti, Two Applications of Logic to Mathematics, Iwanami and Princeton University Press, Tokyo and Princeton (1978).

[11] G.Takeuti, Quantum Set Theory, Current Issues in Quantum Logic, eds. E.Beltrametti and B.C.van Frassen, Plenum,New York (1981) pp.303-322

[12] S.Titani, Lattice Valued Set Theory, Arch. Math. Logic 38-6(1999) pp.395-421

[13] S.Titani, A completeness Theorem of Quantum Set Theory, to appear in Handbook of Quantum Logic and Quantum Structures : Quantum Logic, eds.K.Engesser, D.M.Gabbay and D.Lehmann, Elsevier Science Ltd.